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A new multiple Dirichlet series induced by a higher order form

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1 Introduction

This note proposes a new link between two relatively new objects in Number Theory, higher-order automorphic forms and multiple Dirichlet series.

First-order automorphic forms of weight $k \in 2\mathbb{Z}_+$ for a lattice Γ in $\mathrm{PSL}_2(\mathbb{R})$ are defined as smooth complex-valued functions f on the upper-half plane $\mathfrak{H}$ such that

- $f|_k(\gamma - 1)$ is a modular form of weight k for Γ
- $f|_k\pi = f$, for every parabolic element of Γ
- f has a "moderate growth at the cusps".

Here the action $|_k$ of $\mathrm{PSL}_2(\mathbb{R})$ on functions $g : \mathfrak{H} \mapsto \mathbb{C}$ is defined by

$$(g|_k\gamma)(z) = g(\gamma z)(cz + d)^{-k}$$

with $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ in $\mathrm{PSL}_2(\mathbb{R})$. We extend the action to the group ring $\mathbb{C}[PSL_2(\mathbb{R})]$ by linearity. This definition can be extended in a natural way to higher-order forms. (Note that the order in this definition differs from that in the definitions given in previous papers in the subject. The reason for the modification of terminology is related to the fact that, by Fourier transform, $\mathbb{Z}$-supported tempered distributions are mapped to higher order $\mathbb{Z}$-invariants, where the natural differentiation order of the distributions corresponds to the new notion of order.)

Though some of the ideas behind the investigation of multiple Dirichlet series originated earlier, the systematic study began in the mid-90's ([3], [6] etc.) A definition of *multiple Dirichlet series* is given in [6]:

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{1}{m_1^{s_1} \dots m_n^{s_n}} \int_0^{\infty} \cdots \int_0^{\infty} \frac{a(m_1, \dots, m_n, t_1, \dots, t_l)}{t_1^{w_1} \dots t_l^{w_l}} dt_1 \dots dt_l \quad (1)$$

where $a(m_1, \dots, m_n, t_1, \dots, t_l)$ is a complex-valued smooth function, or, more generally, vectors with entries such series. Among these series, those that have a meromorphic continuation to the entire $\mathbb{C}^n$ and satisfy enough functional equations are of particular interest for applications and are sometimes refered to as 'perfect'. Constructing 'perfect' multiple Dirichlet series is much harder than the corresponding problem in classical Dirichlet series and it is one of the main aims of the theory. Apart from the multiple Dirichlet series obtained from metaplectic Eisenstein series, essentially none of the known 'perfect' multiple Dirichlet series are constructed as Mellin transforms but by other techniques, cf. [1], [2], [5], etc. In this note, we construct a perfect multiple Dirichlet series as the Mellin transform of a first-order form. This first-order form is essentially the Eisenstein series twisted by modular symbols. Although the resulting double Dirichlet series has infinitely many poles, and is thus not as suitable for current application as it would have been if it had finitely many poles, it is, to our knowledge, the first example of a non-classical modular object producing a double Dirichlet series via a Mellin transform. This may suggest that there may be a broader class of objects generating 'perfect' multiple Dirichlet series in a systematic way similar to that between modular forms and Dirichlet series.

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2 Eisenstein series twisted by modular symbols

Let $\Gamma \subset PSL_2(\mathbb{R})$ be a non-uniform lattice. As usual we write $x + iy = z \in \mathfrak{H}$. Fix a set $\{\mathfrak{a}_1, \dots, \mathfrak{a}_m\}$ of representatives of the inequivalent cusps of the group Γ. For each $\mathfrak{a}_j$, we consider a scaling matrix $\sigma_{\mathfrak{a}_j}$ such that $\sigma_{\mathfrak{a}_j}(\infty) = \mathfrak{a}_j$ and

$$\sigma_{\mathfrak{a}_j}{}^{-1}\Gamma_{\mathfrak{a}_j}\sigma_{\mathfrak{a}_j} = \Gamma_\infty = \left\{\pm \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \;\middle|\; m \in \mathbb{Z}\right\}$$

where $\Gamma_{\mathfrak{a}_j}$ is the stabilizer of $\mathfrak{a}_j$ in Γ.

Let $\psi : \Gamma \to \mathbb{C}$ be a group homomorphism which is zero on all parabolic elements. For every $k \in 2\mathbb{Z}_+$ and $\mathfrak{a} \in \{\mathfrak{a}_1, \dots, \mathfrak{a}_m\}$, we set

$$E_{\mathfrak{a}}(z, s, k; \psi) := \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \psi(\gamma) \mathrm{Im}(\sigma_{\mathfrak{a}}{}^{-1}\gamma z)^s j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k},$$

where $j(\gamma, z) = cz + d$ when $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

This series is absolutely convergent for $\mathrm{Re}(s) > 2 - \frac{k}{2}$ and for $k = 0$ it can be meromorphically continued to all of $\mathbb{C}$ ([8]). It further satisfies

$$E_{\mathfrak{a}}(\gamma z, s, k; \psi) j(\gamma, z)^{-k} = E_{\mathfrak{a}}(z, s, k; \psi) + \psi(\gamma^{-1}) E_{\mathfrak{a}}(z, s, k) \tag{2}$$

where

$$E_{\mathfrak{a}}(z, s, k) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}$$

is the classical Eisenstein series at $\mathfrak{a}$.

The function $E_{\mathfrak{a}}(z,s,k;\psi)$ is a weight k first-order automorphic form for each s for which it is defined (cf. [4], where first order is called second-order according to an older convention) and it has been used to obtain information about the distribution of modular symbols ([7], [11]), the number of appearances of a given generator in reduced words of Γ etc.

Consider now the lattice

$$\Gamma^* = \langle \Gamma_0(N), W_N \rangle = \Gamma_0(N) \cup \Gamma_0(N) W_N$$

where $\Gamma_0(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbf{Z}); N|c\}$ and $W_N = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}$.

We now concentrate on the case $k=0$ for simplicity and because the weight does not affect the point we want to make. Let f be a newform of weight 2 for $\Gamma_0(N)$ such that $L_f(1)=0$. We set

$$\psi(\gamma) = \langle f, \gamma \rangle := \int_{i\infty}^{\gamma i\infty} f(z)dz$$

and $E_{\mathfrak{a}}(z,s;f) := E_{\mathfrak{a}}(z,s;\psi)$. The Fourier expansion of $E_{\mathfrak{a}}(z,s;f)$ at $\mathfrak{b}$ is

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z,s;f) = \phi_{\mathfrak{a}\mathfrak{b}}(s;f) y^{1-s} + \sum_{n \neq 0} \phi_{\mathfrak{a}\mathfrak{b}}(n,s;f) W_s(nz)$$

with

$$\phi_{\mathfrak{a}\mathfrak{b}}(s;f) = \pi \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{c \in C_{\mathfrak{a}\mathfrak{b}}} \frac{S_{\mathfrak{a}\mathfrak{b}}(0,0,f;c)}{c^{2s}}$$

and

$$\phi_{\mathfrak{a}\mathfrak{b}}(n,s;f) = \frac{\pi^s}{\Gamma(s)} |n|^{s-1} \sum_{c \in C_{\mathfrak{a}}} \frac{S_{\mathfrak{a}\mathfrak{b}}(n,0,f;c)}{c^{2s}}$$

where $C_{\mathfrak{a}\mathfrak{b}} = \{c>0; \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \in {\sigma_{\mathfrak{a}}}^{-1} \Gamma^* \sigma_{\mathfrak{b}}\}$ and

$$S_{\mathfrak{a}\mathfrak{b}}(m,n,f;c) = \sum_{\substack{\gamma \in \Gamma_\infty \backslash {\sigma_{\mathfrak{a}}}^{-1} \Gamma^* \sigma_{\mathfrak{b}} / \Gamma_\infty \\ \gamma_c = c}} < f, \sigma_{\mathfrak{a}} \gamma {\sigma_{\mathfrak{b}}}^{-1} > e^{2\pi i (n \frac{\gamma_a}{c} + m \frac{\gamma_d}{c})}.$$

We denote the matrix $(\phi_{\mathfrak{a}\mathfrak{b}}(s;f))$ by $\Phi(s;f)$.

Set

$$\mathbf{E}(z,s;f) := (E_{\mathfrak{a}_1}(z,s;f), \ldots, E_{\mathfrak{a}_m}(z,s;f))^T$$

and

$$\mathbf{E}(z,s) := (E_{\mathfrak{a}_1}(z,s), \ldots, E_{\mathfrak{a}_m}(z,s))^T$$

where ^{T} indicates matrix transpose. In ([10]), it is proved that $\mathbf{E}(z,s;f)$ can be meromorphically continued in s and that, for all s for which $\Phi(s,f)$, $\Phi(1-s)$ and $\Phi(s)$ are defined,

$$\Phi(s)\mathbf{E}(z,1-s;f) = \mathbf{E}(z,s;f) - \Phi(s,f)\Phi(1-s)\mathbf{E}(z,s). \tag{3}$$

$\Phi(s) = (\phi_{\mathfrak{a}\mathfrak{b}}(s))$ is the scattering matrix of the standard Eisenstein series and satisfies

$$\Phi(s)\Phi(1-s;f) = -\Phi(s,f)\Phi(1-s) \tag{4}$$

3 L-functions of $\mathbf{E}$

It is possible to use (4) to obtain a standard functional equation in s for $\mathbf{E}(z,s;f)$ (i.e. not "shifted" by a multiple of $\mathbf{E}(z,s)$). Set

$$\tilde{\mathbf{E}}(z,s;f) := \mathbf{E}(z,s;f) - \frac{1}{2}\Phi(s,f)\Phi(1-s)\mathbf{E}(z,s).$$

Then, using the functional equation $\Phi(s)\mathbf{E}(z,1-s) = \mathbf{E}(z,s)$, we obtain

$$\begin{aligned}\Phi(s)\tilde{\mathbf{E}}(z,1-s;f) &= \Phi(s)\mathbf{E}(z,1-s;f) - \frac{1}{2}\Phi(s)\Phi(1-s,f)\Phi(s)\mathbf{E}(z,1-s)\\ &= \mathbf{E}(z,s;f) - \Phi(s,f)\Phi(1-s)\mathbf{E}(z,s) - \frac{1}{2}\Phi(s)\Phi(1-s,f)\Phi(s)\mathbf{E}(z,1-s)\\ &= \mathbf{E}(z,s;f) - \Phi(s,f)\Phi(1-s)\mathbf{E}(z,s) + \frac{1}{2}\Phi(s,f)\Phi(1-s)\mathbf{E}(z,s) = \tilde{\mathbf{E}}(z,s;f).\end{aligned} \tag{5}$$

$\tilde{\mathbf{E}}(z,s;f)$ has infinitely many poles at $\mathrm{Re}(s)=\frac{1}{2}$ and, possibly, finitely many poles in $(1/2,1)$ because $\mathbf{E}(z,s)$ has no poles on $\mathrm{Re}(s)=\frac{1}{2}$.

In terms of z, (2) and $L_f(1)=0$ imply that

$$\mathbf{E}(W_N z,s;f) = \mathbf{E}(z,s;f) + \langle f, W_N\rangle \mathbf{E}(z,s) = \mathbf{E}(z,s;f)$$

and thus

$$\tilde{\mathbf{E}}(W_N z,s;f) = \tilde{\mathbf{E}}(z,s;f). \tag{6}$$

We next define the "completed" L-function of $\tilde{E}_{\mathfrak{a}}$. For every w for which $\tilde{E}_{\mathfrak{a}}(iy,w;f)$ is defined and for $\mathrm{Re}(s) >\max(1+w,2-w)$, set

$$\tilde{\Lambda}_{\mathfrak{a}}(s,w) = \int_0^\infty \left(\tilde{E}_{\mathfrak{a}}(iy,w;f) - a_{\mathfrak{a}}(w)y^w - b_{\mathfrak{a}}(w)y^{1-w}\right) y^s \frac{dy}{y}$$

where $a_{\mathfrak{a}}(w)y^w + b_{\mathfrak{a}}(w)y^{1-w}$ is the constant term of $\tilde{E}_{\mathfrak{a}}(z,w;f)$. We also set

$$\tilde{\mathbf{\Lambda}} := (\tilde{\Lambda}_{\mathfrak{a}_1}, \dots, \tilde{\Lambda}_{\mathfrak{a}_m}).$$

We shall prove

Theorem 3.1. *The function $\tilde{\mathbf{\Lambda}}(s,w)$ is a (vector-valued) double Dirichlet series which can be meromorphically continued to $\mathbb{C}^2$. If $\tilde{\mathbf{E}}(z,w;f)$ does not have a pole at w_0, then $\tilde{\mathbf{\Lambda}}(s,w)$ has a pole at $s=\pm w_0$, and $s=\pm(w_0-1)$. It further satisfies the functional equations*

$$N^s\tilde{\mathbf{\Lambda}}(s,w) = \tilde{\mathbf{\Lambda}}(-s,w).$$

and

$$\Phi(w)\tilde{\mathbf{\Lambda}}(s,1-w) = \tilde{\mathbf{\Lambda}}(s,w)$$

Proof. (6) implies that

$$\tilde{\Lambda}_{\mathfrak{a}}(s,w) = \int_{1/\sqrt{N}}^{\infty} \left(\tilde{E}_{\mathfrak{a}}(iy,w;f) - a_{\mathfrak{a}}(w)y^{w} - b_{\mathfrak{b}}(w)y^{1-w} \right) y^{s}\frac{dy}{y} +$$

$$\int_{1/\sqrt{N}}^{\infty} \left(\tilde{E}_{\mathfrak{a}}(i/(Ny),w;f) - a_{\mathfrak{a}}(w)(Ny)^{-w} - b_{\mathfrak{b}}(w)(Ny)^{w-1} \right) (Ny)^{-s}\frac{dy}{y} =$$

$$\int_{1/\sqrt{N}}^{\infty} \left(\tilde{E}_{\mathfrak{a}}(iy,w;f) - a_{\mathfrak{a}}(w)y^{w} - b_{\mathfrak{b}}(w)y^{1-w} \right) (y^{s} + N^{-s}y^{-s})\frac{dy}{y} +$$

$$N^{-s}\int_{1/\sqrt{N}}^{\infty} \left(a_{\mathfrak{a}}(w)y^{w} + b_{\mathfrak{a}}(w)y^{1-w} - a_{\mathfrak{a}}(w)(Ny)^{-w} - b_{\mathfrak{a}}(w)(Ny)^{w-1} \right) y^{-s}\frac{dy}{y} =$$

$$\int_{1/\sqrt{N}}^{\infty} \left(\tilde{E}_{\mathfrak{a}}(iy,w;f) - a_{\mathfrak{a}}(w)y^{w} - b_{\mathfrak{b}}(w)y^{1-w} \right) (y^{s} + N^{-s}y^{-s})\frac{dy}{y} +$$

$$\frac{a_{\mathfrak{a}}(w)(\sqrt{N})^{-s-w}}{s-w} + \frac{b_{\mathfrak{a}}(w)(\sqrt{N})^{w-1-s}}{s+w-1} - \frac{a_{\mathfrak{a}}(w)(\sqrt{N})^{-s-w}}{s+w} - \frac{b_{\mathfrak{a}}(w)(\sqrt{N})^{-s+w-1}}{s-w+1}$$

Since for every w for which $\tilde{E}_{\mathfrak{a}}(iy,w;f)$ is defined, $E_{\mathfrak{a}}(iy,w;f)-$constant term$=O(e^{-\pi y})$, and the analogous fact holds for $E_{\mathfrak{a}}(z,w)$, the last integral is well-defined and gives a holomorphic function in s. This shows that $\tilde{\Lambda}_{\mathfrak{a}}$ can be meromorphically continued to $\mathbb{C}^2$ with poles at $s = \pm w$ and $w-1 = \pm s$, and that $N^{s}\tilde{\Lambda}_{\mathfrak{a}}(s,w) = \tilde{\Lambda}_{\mathfrak{a}}(-s,w)$.

Further, since the vector of constant terms of the entries of $\tilde{\mathbf{E}}(z,s;f)$ satisfies the same functional equation in s as $\tilde{\mathbf{E}}(z,s;f)$ (that is, (5)), we immediately deduce that $\tilde{\mathbf{\Lambda}}(s,w)$ satisfies

$$\Phi(w)\tilde{\mathbf{\Lambda}}(s,1-w) = \tilde{\mathbf{\Lambda}}(s,w) \tag{7}$$

Finally, by the formulas for the Fourier coefficients of $E_{\mathfrak{a}}(z,s;f)$, $E_{\mathfrak{a}}(z,s)$, and for the functions $\phi_{\mathfrak{ab}}(s), \phi_{\mathfrak{ab}}(s,f)$ we observe that $\tilde{\Lambda}_{\mathfrak{a}}(s,w)$ is a double Dirichlet series according the definition in the introduction.

Then the functional equations just proved imply the result. □

Remark. Incidentally, the fact that $\tilde{\mathbf{E}}(z,w;f)$, and thus $\tilde{\mathbf{\Lambda}}(s,w)$, has infinitely many poles in w ([9]) shows that $\tilde{\mathbf{\Lambda}}$ is a 'genuine' double Dirichlet series and not a finite sum of products of (one-variable) L-functions of classical modular forms.

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