Quantum fluctuations and correlations in open quantum Dicke models

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In the vicinity of ground-state phase transitions quantum correlations can display non-analytic behavior and critical scaling. This signature of emergent collective effects has been widely investigated within a broad range of equilibrium settings. However, under nonequilibrium conditions, as found in open quantum many-body systems, characterizing quantum correlations near phase transitions is challenging. Moreover, the impact of local and collective dissipative processes on quantum correlations is not broadly understood. Here we consider as a paradigmatic setting the superradiant phase transition of the open quantum Dicke model and characterize quantum and classical correlations across the phase diagram. We develop an approach to quantum fluctuations which allows us to show that local dissipation, which cannot be treated within the commonly employed Holstein-Primakoff approximation, rather unexpectedly leads to an enhancement of collective quantum correlations, and to the emergence of a nonequilibrium superradiant phase in which the bosonic and spin degrees of freedom of the Dicke model are entangled.

I. INTRODUCTION

The Dicke model [1] provides a paradigmatic framework for the study of the interaction between a large ensemble of atoms, described by N spin-1/2 (two-level) particles, and an electromagnetic cavity field, described by a bosonic mode [cf. Fig. 1(a)]. This model has been thoroughly investigated in equilibrium [2–8], where it displays a second-order ground-state transition from a normal to a superradiant phase. While the order-parameter behavior is captured by a mean-field treatment [9], studying quantum correlations requires the analysis of quantum fluctuations [10–17]. In equilibrium, this is typically done within the so-called Holstein-Primakoff approximation [18]. This exploits that the system Hamiltonian can be written in terms of collective (macroscopic) spin operators, which approximately behave as bosons when the system is close to its ground-state.

Nowadays, also due to a debate concerning a no-go theorem on the experimental realization of the equilibrium Dicke model [19–23], investigations focus on the stationary state of the open Dicke model [cf. Fig. 1(a)] rather than on the equilibrium state of its closed version [2–7]. Open quantum Dicke models feature a *nonequilibrium* superradiant phase transition, see Fig. 1(b), which is exactly captured by a mean-field approach [24, 25]. However, in these settings, analyzing quantum fluctuations is challenging [9, 26–29]. As a consequence, little is known about correlations in the nonequilibrium stationary states of open quantum Dicke models [29–31] — whose behavior is expected to be substantially different from that in the equilibrium ground state [12] — and even less about the role of local dissipative processes, such as local spin-decay [29, 32, 33] [cf. Fig. 1(a)].

In this paper, we provide a complete characterization of quantum and classical correlations in open quantum Dicke models. We achieve this by developing an approach

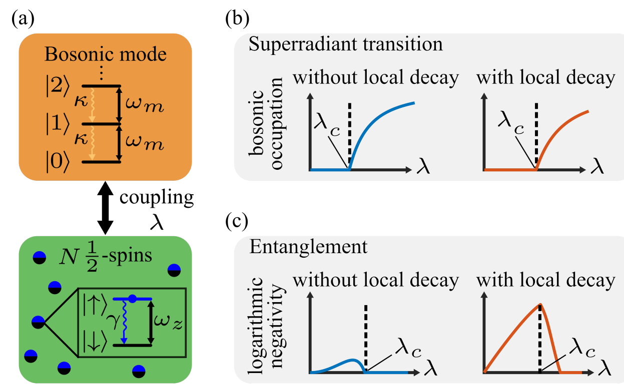

FIG. 1. **Open quantum Dicke model: superradiant phase transition and entanglement.** (a) An ensemble of N spin-1/2 systems (with energy splitting ω_z between up-state $|\uparrow\rangle$ and down-state $|\downarrow\rangle$) is coupled to a bosonic mode (the frequency ω_m determines the energy-cost of creating one field excitation $|n\rangle \to |n+1\rangle$). The presence of an environment induces boson losses (at rate κ) and local spin-decay (at rate γ). We will consider all dynamical parameters in units of κ. (b) At a critical coupling strength $\lambda = \lambda_c$, the system undergoes a superradiant phase transition, characterized by a macroscopic occupation ($\propto N$) of the bosonic mode, both in the presence and in the absence of local spin-decay. (c) The presence of local spin-decay leads to stronger quantum correlations and also "stabilizes" an entangled nonequilibrium superradiant phase.

to quantum correlations based on the theory of quantum fluctuation operators [34–40]. The idea is to focus on the Heisenberg equations for quantum fluctuations rather than on the bosonization of the dynamical generator. This allows one to treat settings in which the dynamics is not of collective type and where the commonly used Holstein-Primakoff transformation cannot be exploited, e.g. in the presence of local dissipative terms. We investigate various correlation measures, such as quantum discord and classical correlation, and show that they display non-analytic behavior at the critical coupling strength [see e.g. Fig. 1(c)]. Furthermore, we analyze bipartite entanglement between the spins and the bosonic mode. We find that the presence of local spin-decay — an unavoidable process in experiments which is usually considered detrimental for quantum effects — is unexpectedly beneficial for the build-up of quantum correlations. Our results indicate that this process leads to increased entanglement in the normal phase, and to the emergence of a nonequilibrium superradiant phase [cf. Fig. 1(c)] where entanglement is nonvanishing.

The paper is organized as follows. In Sec. II we introduce the open Dicke model subject to our investigation and in Sec. III we review its non-equilibrium superradiant phase transition. Building on these results, in Sec. IV we introduce quantum fluctuation operators and study their correlations in Sec. V. We discuss our results in Sec. VI. In Appendix A we present details on the analysis of Sec. III, while Appendices B and C contain details on the derivation of the dynamics of the covariance matrix of quantum fluctuations.

II. OPEN QUANTUM DICKE MODEL

The Dicke model consists of an ensemble of N spin-1/2 subsystems collectively interacting with a single bosonic mode, see Fig. 1(a). Spin operators for the kth particle are denoted as σ_k^α, with $\sigma^x = (|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|)/2$, $\sigma^z = (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)/2$, and $\sigma^y = -2i\sigma^z\sigma^x$. Here, the states $|\uparrow\rangle$, $|\downarrow\rangle$ are the single-particle spin states. The bosonic mode is described by creation and annihilation operators $a^\dagger$ and a, respectively. For later convenience, we also introduce the spin operators $\sigma^\pm = \sigma^x \pm i\sigma^y$ and the bosonic quadrature operators $q = i(a - a^\dagger)/\sqrt{2}$ and $p = (a + a^\dagger)/\sqrt{2}$.

The system is governed by a Markovian open quantum dynamics, under which the time-evolution of an operator O follows the Lindblad equation $\dot{O}(t) = \mathcal{L}_N[O(t)]$ [41–43] with generator

$$\mathcal{L}_N[O] := i[H_N^D, O] + \kappa\left(a^\dagger O a - \frac{1}{2}\{a^\dagger a, O\}\right) + \gamma\sum_{k=1}^N\left(\sigma_k^+ O\sigma_k^- - \frac{1}{2}\{\sigma_k^+\sigma_k^-, O\}\right). \quad (1)$$

The first term on the right-hand side of Eq. (1) gives the coherent contribution to the dynamics implemented by the Dicke Hamiltonian (setting $\hbar = 1$)

$$H_N^D = \omega_m a^\dagger a + \omega_z S^z + \frac{2\lambda}{\sqrt{N}}(a + a^\dagger)S^x. \quad (2)$$

Here, $\omega_m > 0$ is the bosonic mode frequency, $\omega_z > 0$ the energy splitting between spin states and $\lambda > 0$ the coupling parameter [cf. Fig. 1(a)]. The Dicke Hamiltonian (2) is written in terms of the collective operators $S^\alpha = \sum_{k=1}^N \sigma_k^\alpha$, obeying $[S^\alpha, S^\beta] = i\epsilon^{\alpha\beta\gamma}S^\gamma$, where $\epsilon^{\alpha\beta\gamma}$ is the Levi-Civita symbol. Summation over repeated indices is implied here and in the following. The factor $1/\sqrt{N}$, which rescales the collective spin-boson coupling in H_N^D, is necessary for a well-defined thermodynamic limit [9]. The last two terms in Eq. (1) account for irreversible dynamical effects. These are decay of bosonic excitations at rate κ as well as local (individual) spin-decay, $|\uparrow\rangle \to |\downarrow\rangle$, at rate γ. As becomes clear from Eq. (1), the latter process is not described through collective, but rather local, spin (jump) operators σ_k^-.

We note that the above open quantum Dicke model is not just of mere theoretical interest but actually finds application in the description of several experimental settings [44, 45].

III. SUPERRADIANT TRANSITION

The open quantum Dicke model undergoes a phase transition — as a function of the coupling strength λ — from a normal stationary phase, with subextensive (in N) bosonic occupation, to a superradiant one where the bosonic mode becomes macroscopically occupied [9, 24, 25] [see sketch in Fig. 1(b)]. An order parameter for this transition is the stationary expectation of the renormalized number operator $a^\dagger a/N$ in the thermodynamic limit of large number of spins, $N \to \infty$.

The form of the order parameter suggests the definition of the so-called *mean-field operators*

$$m_N^q = \frac{q}{\sqrt{N}}, \quad m_N^p = \frac{p}{\sqrt{N}} \quad \text{and} \quad m_N^\alpha = \frac{S^\alpha}{N}, \quad (3)$$

where the last term must be considered for $\alpha = x, y, z$. The first two operators are relevant as they provide the order parameter through $(m_N^q)^2 + (m_N^p)^2 - 1/N = 2a^\dagger a/N$. The mean-field operators of the spin ensemble [last term in Eqs. (3)] are also important for studying the model. Indeed, by computing the action of the generator $\mathcal{L}_N$ in Eq. (1) on the mean-field operators in Eqs. (3), one finds that all these operators are dynamically coupled. Denoting the second and the third term (dissipators) in Eq. (1) by $\mathcal{D}_N^\kappa[O]$ and $\mathcal{D}_N^\gamma[O]$, respectively, we can compute the contribution of each term in Eq. (1) to $\dot{m}_N^\alpha(t)$ (with $\alpha = x, y, z, q, p$) separately. The coherent part acts non-trivially on all mean-field operators since

$$i[H_N^D, m_N^\alpha] = -\omega_z\epsilon^{z\alpha\gamma}m_N^\gamma - 2^{3/2}\lambda m_N^p\epsilon^{x\alpha\gamma}m_N^\gamma, \quad (4)$$

for $\alpha = x, y, z$, and

$$\begin{aligned} i[H_N^D, m_N^q] &= \omega_m m_N^p + 2^{3/2}\lambda m_N^x, \\ i[H_N^D, m_N^p] &= -\omega_m m_N^q. \end{aligned} \quad (5)$$

For the dissipator $\mathcal{D}_N^\kappa$, however, we immediately see that $\mathcal{D}_N^\kappa[m_N^\alpha] = 0$ for $\alpha = x, y, z$. The remaining two components are transformed linearly

$$\mathcal{D}_N^{\kappa}[m_N^q] = -\frac{\kappa}{2} m_N^q, \quad \mathcal{D}_N^{\kappa}[m_N^p] = -\frac{\kappa}{2} m_N^p. \tag{6}$$

The dissipator $\mathcal{D}_N^{\gamma}$ yields $\mathcal{D}_N^{\gamma}[m_N^q] = 0$, $\mathcal{D}_N^{\gamma}[m_N^p] = 0$ and it acts on the mean-field operators for $\alpha = x, y, z$ as

$$\mathcal{D}_N^{\gamma}[m_N^{\alpha}] = -\frac{\gamma}{2}\big(\delta^{x\alpha} m_N^x + \delta^{y\alpha} m_N^y + \delta^{z\alpha}(\mathbb{1} + 2m_N^z)\big). \tag{7}$$

In the thermodynamic limit, the time-evolved operators $m_N^{\alpha}(t)$ ($\alpha = x, y, z, q, p$) behave as multiples of the identity proportional to their expectation, i. e. $m_N^{\alpha}(t) \to m^{\alpha}(t) = \lim_{N\to\infty} \langle m_N^{\alpha}(t) \rangle$ [24, 25]. Hence, exploiting Eqs. (4-7), it is possible to show that they obey the differential equations (we drop the explicit time-dependence)

$$\begin{aligned}
\dot{m}^x &= -\omega_z m^y - \frac{\gamma}{2} m^x\,, \\
\dot{m}^y &= \omega_z m^x - 2^{3/2}\lambda m^p m^z - \frac{\gamma}{2} m^y\,, \\
\dot{m}^z &= 2^{3/2}\lambda m^p m^y - \frac{\gamma}{2}(1 + 2m^z)\,, \\
\dot{m}^q &= \omega_m m^p + 2^{3/2}\lambda m^x - \frac{\kappa}{2} m^q\,, \\
\dot{m}^p &= -\omega_m m^q - \frac{\kappa}{2} m^p\,.
\end{aligned} \tag{8}$$

These equations feature two different stationary regimes, separated by a critical value of the coupling strength

$$\lambda_c = \sqrt{\frac{\left[\omega_m^2 + \left(\frac{\kappa}{2}\right)^2\right]\left[\omega_z^2 + \left(\frac{\gamma}{2}\right)^2\right]}{4\omega_z\omega_m}}\,. \tag{9}$$

This can be seen by solving the system of equations obtained by setting the time derivatives in Eqs. (8) to zero. If $\gamma > 0$, one finds that for $\lambda > \lambda_c$ there are two stable stationary solutions

$$\begin{aligned}
m^x &= \pm\frac{\omega_z}{\sqrt{2(\omega_z^2 + (\frac{\gamma}{2})^2)}}\frac{\lambda_c}{\lambda}\sqrt{1 - \frac{\lambda_c^2}{\lambda^2}} \\
m^y &= \mp\frac{\gamma}{2^{3/2}\sqrt{\omega_z^2 + (\frac{\gamma}{2})^2}}\frac{\lambda_c}{\lambda}\sqrt{1 - \frac{\lambda_c^2}{\lambda^2}} \\
m^z &= -\frac{1}{2}\frac{\lambda_c^2}{\lambda^2} \\
m^q &= \pm\frac{\kappa\omega_z\lambda_c}{(\omega_m^2 + (\frac{\kappa}{2})^2)\sqrt{\omega_z^2 + (\frac{\gamma}{2})^2}}\sqrt{1 - \frac{\lambda_c^2}{\lambda^2}} \\
m^p &= \mp\frac{2\omega_z\lambda_c}{(\omega_m + (\frac{\kappa}{2})^2\frac{1}{\omega_m})\sqrt{\omega_z^2 + (\frac{\gamma}{2})^2}}\sqrt{1 - \frac{\lambda_c^2}{\lambda^2}}
\end{aligned} \tag{10}$$

and an unstable one

$$m^z = -\frac{1}{2}, \quad m^x = m^y = m^q = m^p = 0. \tag{11}$$

The latter is however the only stable solution for $0 < \lambda < \lambda_c$. Considering instead $\gamma = 0$, the two stable stationary solutions in Eqs. (10) have to be replaced by

$$\begin{aligned}
m^x &= \pm\frac{\sqrt{1 - \frac{\lambda_c^4}{\lambda^4}}}{2} \\
m^y &= 0 \\
m^z &= -\frac{\lambda_c^2}{2\lambda^2} \\
m^q &= \pm\frac{\kappa}{\sqrt{2}}\frac{\lambda}{(\omega_m^2 + (\frac{\kappa}{2})^2)}\sqrt{1 - \frac{\lambda_c^4}{\lambda^4}} \\
m^p &= \mp\frac{\lambda}{(\omega_m + (\frac{\kappa}{2})^2\frac{1}{\omega_m})/\sqrt{2}}\sqrt{1 - \frac{\lambda_c^4}{\lambda^4}}.
\end{aligned} \tag{12}$$

In this case one also has additional stationary solutions, obtained by sending $m^z \to -m^z$ in Eqs. (11) and (12), which are, however, not stable. To summarize, as shown in Appendix A, for $0 < \lambda < \lambda_c$ there exists a unique stationary solution to the system in Eqs. (8) which is asymptotically stable, given by the trivial solution in Eq. (11). This is the normal phase. For $\lambda > \lambda_c$, this becomes unstable but two other stable solutions emerge which spontaneously break the symmetry $a \to -a$, $\sigma^- \to -\sigma^-$ of the generator $\mathcal{L}_N$ [9, 46]. These are given in Eqs. (10) for $\gamma > 0$ and in Eqs. (12) for $\gamma = 0$. Their finite stationary values of $m^{x/p}$ imply a macroscopic occupation of the bosonic mode in this superradiant phase.

IV. QUANTUM FLUCTUATIONS

The observables m_N^{α} — which are analogous to the sample mean variables of the law of large numbers — provide, in the thermodynamic limit $N \to \infty$, a classical average description of the Dicke model [25, 39, 47]. The latter carries no information about correlations or fluctuations. In order to explore collective quantum correlations in the two phases of the model, it is necessary to introduce a new set of observables, the so-called quantum *fluctuation operators* [34–40]. These account for deviations of the operators m_N^{α} from their average behavior and are analogous to the fluctuation variables of central limit theorems.

Quantum fluctuation operators are defined as

$$F_N^{\alpha} = \sqrt{N}\left(m_N^{\alpha} - \langle m_N^{\alpha}\rangle\right)\,. \tag{13}$$

For $\alpha = x, y, z$, these are the usual spin fluctuation operators [38, 39], and we have defined the bosonic ones ($\alpha = q, p$) in full analogy. Remarkably, despite being collective, the operators in Eqs. (13) retain a quantum character in the thermodynamic limit, in which the limiting operators $F^{\alpha} = \lim_{N\to\infty} F_N^{\alpha}$ behave as bosons (for a rigorous discussion see e. g. Ref. [37]). This is straightforward to check for $F^{q/p}$, since $F_N^q = q - \langle q \rangle$ and $F_N^p = p - \langle p \rangle$. However, also collective spin fluctuations give rise to an emergent bosonic mode. This can be seen as follows. Looking at the commutator of fluctuation operators, one

finds that $[F^\alpha, F^\beta] = i\epsilon^{\alpha\beta\gamma} m^\gamma$ $(\alpha, \beta = x, y, z)$, which is a multiple of the identity. Now, we rotate the reference frame for the spin ensemble aligning the z-direction with the direction identified by mean-field variables, which is $\hat{n} = \vec{m}^s/|\vec{m}^s|$ with $\vec{m}^s = (m^x, m^y, m^z)^T$. In this rotated frame we have $\tilde{m}^x = \tilde{m}^y = 0$ and $\tilde{m}^z > 0$, so that the only nonzero commutator is $[\tilde{F}^x, \tilde{F}^y] = i\tilde{m}^z$. Here the corresponding rotation matrix $R_{\hat{n}}$ transforms the vector of fluctuations as $\vec{F} \to \tilde{\vec{F}} = R_{\hat{n}}\vec{F}$. A *canonical* bosonic mode is finally obtained by applying a further rescaling matrix J only changing $\tilde{F}^x \to Q = \tilde{F}^x/\sqrt{\tilde{m}^z}$, $\tilde{F}^y \to P = \tilde{F}^y/\sqrt{\tilde{m}^z}$, which then fulfill $[Q, P] = i$.
In what follows, we work with the set of fluctuations $r = (Q, P, \tilde{F}^z, F^q, F^p)^T$. The first two elements represent an emergent bosonic mode describing collective properties of the spin ensemble; the last two are the fluctuations of the original bosonic mode, while $\tilde{F}^z$ is a fluctuation operator which commutes with the others [37, 39].

To analyze correlations in the Dicke model through fluctuation operators, we introduce the covariance matrix $\tilde{\Sigma}^{\alpha\beta} = \langle\{r^\alpha, r^\beta\}\rangle/2$. Recalling Eq. (13), we can identify the elements $\tilde{\Sigma}^{\alpha\alpha}$ as the susceptibility of the order parameter m^α. For Gaussian states, this matrix contains the full information about fluctuations and can be used to quantify collective correlations [48, 49]. Before going to that, however, we briefly discuss the time-evolution of $\tilde{\Sigma}$ under the dynamics implemented by the generator in Eq. (1). A detailed analysis can be found in Appendix B. For each parameter regime, we consider the dynamics of fluctuations emerging, in the thermodynamic limit, from an initial state which is stationary with respect to the mean-field observables and possesses Gaussian fluctuations. Under this assumption we can infer the dynamics of $\tilde{\Sigma}$ from the one of the covariance matrix $\Sigma^{\alpha\beta} = \langle\{F^\alpha, F^\beta\}\rangle/2$ of quantum fluctuations in the original frame since

$$\dot{\tilde{\Sigma}} = J R_{\hat{n}} \dot{\Sigma} R_{\hat{n}}^T J. \tag{14}$$

Defining the two-point functions $C_N^{\alpha\beta} := \left\langle F_N^\alpha F_N^\beta \right\rangle$ and rewriting the covariance matrix in the original frame in terms of them

$$\Sigma^{\alpha\beta} = \lim_{N\to\infty} \frac{1}{2}\left\langle \{F_N^\alpha, F_N^\beta\}\right\rangle = \lim_{N\to\infty} \frac{C_N^{\alpha\beta} + {C_N^T}^{\alpha\beta}}{2}, \tag{15}$$

we see that in order to arrive at a dynamical equation for Σ the time-derivative of the two-point functions has to be established. Because of their definition, fluctuations are zero averaged, $\langle F_N^\alpha\rangle = 0$, which allows to derive the differential equation

$$\begin{aligned} \dot{C}_N^{\alpha\beta} =& \left\langle \mathcal{L}_N[F_N^\alpha F_N^\beta]\right\rangle \\ =& \langle i F_N^\alpha [H_N^D, F_N^\beta]\rangle + \langle i[H_N^D, F_N^\alpha] F_N^\beta\rangle \\ &+ \langle \mathcal{D}_N^\kappa [F_N^\alpha F_N^\beta]\rangle + \langle \mathcal{D}_N^\gamma [F_N^\alpha F_N^\beta]\rangle. \end{aligned} \tag{16}$$

Evaluating each of the expectation values in the second and third line of Eqs. (16) leads, in the thermodynamic limit, to a matrix differential equation for C and hence to the one for the covariance in the rotated frame [39, 50]

$$\dot{\tilde{\Sigma}}(t) = \tilde{\Sigma}(t)\tilde{G}^T + \tilde{G}\tilde{\Sigma}(t) + \tilde{W}. \tag{17}$$

Here the matrices $\tilde{G}$ and $\tilde{W}$, whose explicit form is given in Appendix B, depend on the parameters of the model and on the stable stationary mean-field variables of Eqs. (8). The time-evolution in Eq. (17) has the structure of a bosonic Gaussian open quantum dynamics [51], suggesting that the Gaussianity of fluctuations is preserved at all times. This is true even in the presence of local decay which does not spoil the Gaussian character of quantum fluctuations [40] (see Appendix B). As we show in Appendix C, Eq. (17) moreover possesses a unique stationary solution $\tilde{\Sigma}_\infty$ as long as $\lambda \neq \lambda_c, 0$. For $\gamma > 0$ this solution is given by (in vectorized form)

$$vec(\tilde{\Sigma}_\infty) = (\tilde{G}\otimes \mathbb{1}_5 + \mathbb{1}_5 \otimes \tilde{G})^{-1} vec(-\tilde{W}).$$

For $\gamma = 0$ a slight modification is required as explained in Appendix C.

Since we are mainly interested in quantum correlations, we discard the information associated with the trivial fluctuation $\tilde{F}^z$. This can be done by extracting from the stationary covariance matrix $\tilde{\Sigma}_\infty$ the 4×4 minor obtained by neglecting its third row and its third column. The resulting covariance matrix

$$\tilde{\Sigma}_\infty^{\mathrm{t-m}} = \begin{pmatrix} \Gamma_s & \Gamma_c \\ \Gamma_c^T & \Gamma_b \end{pmatrix},$$

contains the full information about the two bosonic modes Q, P and F^q, F^p. In particular, Γ_s is the 2×2 matrix containing the second moments of the operators Q, P, Γ_b contains those of F^q, F^p, and Γ_c contains correlations between Q, P and F^q, F^p.

V. QUANTUM AND CLASSICAL CORRELATIONS

In order to explore the correlation structure in the open quantum Dicke model, we focus on measures which can distinguish between correlations of different nature, e. g. quantum or classical, and that are fully determined by the covariance matrix $\tilde{\Sigma}_\infty^{\mathrm{t-m}}$. Since the spin fluctuation operators involve all the spin degrees of freedom, the correlations that we discuss here are of collective type, i. e. reflecting, for instance, how the spin ensemble as a whole is collectively correlated with the bosonic mode.

Firstly, we consider the *classical correlation* $\mathcal{J}$ [52–56] between the spin ensemble and the bosonic mode. For a bipartite quantum state $\rho_{A,B}$ this quantity can be defined as

$$\mathcal{J}(\rho_{A,B}) = S(\rho_A) - \inf_{\{\Pi_i\}} \sum_i p_i S(\mathrm{Tr}_B(\rho_{A,B}\Pi_i)/p_i) \tag{18}$$

with $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$ the von Neumann entropy, $\rho_{A(B)}$ the reduced state for system $A(B)$ and $p_i = \mathrm{Tr}_{A,B}(\rho_{A,B}\Pi_i)$. It encodes the maximum information that can be extracted on one subsystem, by making generalized (Gaussian) measurements (represented by POVMs $\{\Pi_i\}, \sum_i \Pi_i = \mathbb{1}$) on the other one. In this sense, the classical correlation is "asymmetric" since it can be defined in two ways, i. e. either considering that measurements are performed on the spin ensemble or on the bosonic mode. In the following we denote the respective quantities by $\mathcal{J}^{\rightarrow}$ and $\mathcal{J}^{\leftarrow}$. Secondly, we study the so-called *quantum discord* $\mathcal{D}$ [52–56], which is defined as the difference between the total correlation, quantified by the *quantum mutual information*

$$I(\rho_{A,B}) = S(\rho_A) + S(\rho_B) - S(\rho_{A,B}),$$

and the classical correlation $\mathcal{J}$. This quantity measures the genuine quantum contribution to the total correlations between the two subsystems. According to its definition through the classical correlation, also the quantum discord is asymmetric under exchange of the role of the spin ensemble and of the bosonic mode. We adopt the notation for the classical correlation by writing $\mathcal{D}^{\rightarrow}$ for the quantum discord corresponding to measurements on the spin ensemble and $\mathcal{D}^{\leftarrow}$ for the one corresponding to measurements on the bosonic mode. At the level of the covariance matrix $\tilde{\Sigma}^{\mathrm{t-m}}_{\infty}$, explicitly carrying out the minimization in Eq. (18) and defining

$$A = \det(2\Gamma_s), \quad B = \det(2\Gamma_b)$$
$$C = \det(2\Gamma_c), \quad D = \det\big(2\Sigma^{\mathrm{t-m}}_{\infty}\big),$$

leads to the closed expressions for the classical correlation and the quantum discord [52, 54]

$$\mathcal{J}^{\leftarrow}(\tilde{\Sigma}^{\mathrm{t-m}}_{\infty}) = f(\sqrt{A}) - f(\sqrt{E^{\min}}), \tag{19}$$

$$\mathcal{D}^{\leftarrow}(\tilde{\Sigma}^{\mathrm{t-m}}_{\infty}) = f(\sqrt{B}) - f(\nu_-) - f(\nu_+) + f(\sqrt{E^{\min}}), \tag{20}$$

with

$$E^{\min} = \begin{cases} \frac{2C^2+(B-1)(D-A)+2|C|\sqrt{C^2+(B-1)(D-A)}}{(B-1)^2}, \\ \quad \text{for} \quad (D-AB)^2 \leq (1+B)C^2(A+D) \\ \frac{AB-C^2+D-\sqrt{C^4+(-AB+D)^2-2C^2(AB+D)}}{2B}, \\ \quad \text{otherwise} \end{cases}$$

and the function

$$f(x) = \left(\frac{x+1}{2}\right)\log\left[\frac{x+1}{2}\right] - \left(\frac{x-1}{2}\right)\log\left[\frac{x-1}{2}\right].$$

According to Williamson's theorem [57], ν_+, ν_- are the pairwise occurring symplectic eigenvalus of the stationary two-mode covariance matrix, obtained as the diagonal elements of the symplectic diagonalized matrix $2\tilde{\Sigma}^{\mathrm{t-m}}_{\infty}$.

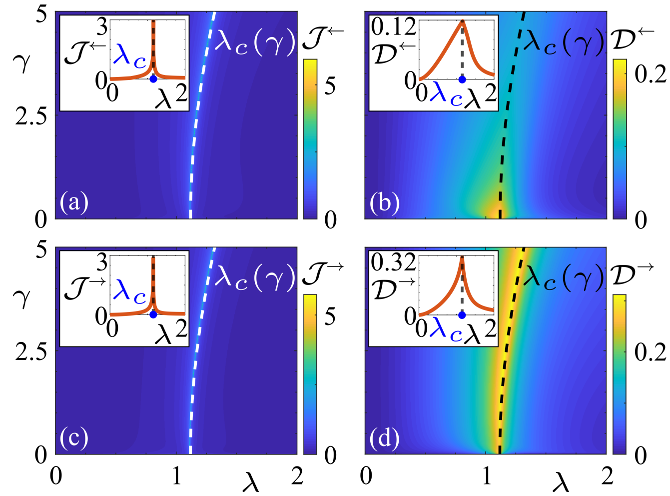

FIG. 2. **Classical correlation and quantum discord.** (a-b) Classical correlation $\mathcal{J}^{\leftarrow}$ and quantum discord $\mathcal{D}^{\leftarrow}$ for measurements on the bosonic mode as functions of γ and λ. (c-d) Classical correlation $\mathcal{J}^{\rightarrow}$ and quantum discord $\mathcal{D}^{\rightarrow}$ for measurements on the spin system as functions of γ and λ. The critical line $\lambda_c(\gamma)$ (dashed line) separates the normal phase from the superradiant one. All quantities display a non-analytic behavior at the critical line, with the classical correlations diverging. The insets visualize the λ-dependence of the corresponding quantities for $\gamma = 2$. For all plots, we fixed $\omega_m = 1$ and $\omega_z = 4$. All dynamical parameters are given in units of κ.

In Fig. 2(a-b), we show the stationary behavior of the classical correlation $\mathcal{J}^{\leftarrow}$ and the quantum discord $\mathcal{D}^{\leftarrow}$, as a function of the coupling strength λ and of the local spin-decay rate γ. As shown, the classical correlation diverges at the nonequilibrium phase transition line, witnessing strong spin-boson correlations. Concerning the presence of quantum correlations, we observe that the quantum discord is different from zero almost everywhere in the phase diagram. It is maximal along the critical line, where it shows a non-analytic behavior even though it remains bounded. We further illustrate the parameter-dependence for $\mathcal{J}^{\rightarrow}$ and $\mathcal{D}^{\rightarrow}$ in Fig. 2(c-d), for which it has to be interchanged $\det(2\Gamma_s) \leftrightarrow \det(2\Gamma_b)$ in Eqs. (19), (20). From there we see that the maxima of the quantum discord are still distributed along the critical line if the measurements are performed on the spin ensemble but they are now of approximately the same height. The classical correlation still diverges at $\lambda_c(\gamma)$.

We now consider the emergence of collective entanglement between the spins and the bosonic mode. This can be quantified from the covariance matrix $\tilde{\Sigma}^{\mathrm{t-m}}_{\infty}$, through the *logarithmic negativity* $\mathcal{E}_{\mathcal{N}}$ — a proper entanglement measure — defined as [58–60]

$$\mathcal{E}_{\mathcal{N}} = \max(0, -\log(\tilde{\nu}_-))\,.$$

Here, $\tilde{\nu}_-$ is the smallest symplectic eigenvalue [60] of the partially transposed covariance matrix obtained from $2\tilde{\Sigma}^{\mathrm{t-m}}_{\infty}$ by exchanging $F^p \to -F^p$ [57, 61–63]. As we

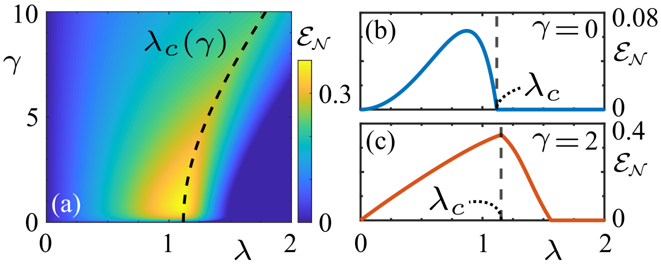

FIG. 3. **Bipartite entanglement between the spin ensemble and the bosonic mode.** (a) Logarithmic negativity $\mathcal{E}_{\mathcal{N}}$ as a function of γ and λ. (b-c) Logarithmic negativity $\mathcal{E}_{\mathcal{N}}$ as a function of λ for $\gamma = 0$ (blue), shown in (b), and $\gamma = 2$ (red), shown in (c). The plots were produced assuming that $\omega_m = 1$ and $\omega_z = 4$. All dynamical parameters are in units of κ.

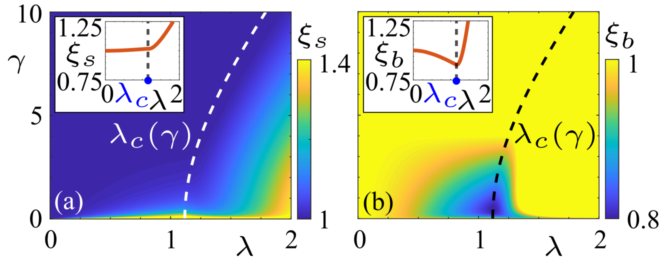

FIG. 4. **Squeezing.** (a) Spin squeezing parameter ξ_s and (b) boson squeezing parameter ξ_b as functions of γ and λ. Both insets show a cut through the density plot at $\gamma = 2$. We have chosen $\omega_m = 1$ and $\omega_z = 4$. All dynamical parameters are in units of κ.

show in Fig. 3(a), the open quantum Dicke model displays collective spin-boson entanglement in a large parameter regime. We are particularly interested in understanding the impact of local spin-decay on entanglement. For small, yet nonvanishing, values of γ we identify a pronounced peak near the critical line $\lambda_c(\gamma)$. This suggests that a small rate of local spin-decay leads to larger entanglement. However, when γ vanishes, entanglement is dramatically reduced. This becomes evident when comparing the behavior of entanglement, as a function of λ, for $\gamma = 0$ and $\gamma \neq 0$. An example is shown in Fig. 3(b-c). In the absence of local spin-decay, entanglement vanishes at the critical point and is always zero in the superradiant phase [cf. Fig. 3(b)]. However, when local spin-decay is present, entanglement assumes larger values across the whole phase diagram and can also persist in the superradiant phase [cf. Fig. 3(c)]. Furthermore, for $\gamma \neq 0$, the logarithmic negativity shows a non-analytic behavior at the critical point and undergoes a "sudden death" well inside the superradiant phase, as shown in Fig. 3(c). These results show that local spin-decay has, rather surprisingly, an overall beneficial effect on quantum correlations, and on quantum entanglement in particular. Comparing Figs. 2(b),(d) and Fig. 3(a), we also see that there exist parameter regions where the quantum discord assumes a finite value but the logarithmic negativity is zero. In this region, the quantum state of fluctuations is separable but nevertheless non-trivially quantum correlated.

Finally, we analyze quantum correlations within each subsystem separately. These are measured by the squeezing parameter $\xi = 2\min(\Theta_1, \Theta_2)$ [50, 64–66] where Θ_1, Θ_2 denote the eigenvalues of Γ_s for spin squeezing and of Γ_b for boson squeezing. The parameter ξ quantifies the minimum variance among all possible quadrature operators. A state is called squeezed if $\xi < 1$, i. e. if the variance in one of the quadratures is smaller than the smallest possible simultaneous uncertainty of two canonically conjugated quadrature operators (also referred to as shot-noise limit [66]). Fig. 4(a) shows that there is no spin-squeezing in the stationary state of the model since ξ_s is always larger than or equal to 1. In contrast, the system can feature squeezing in the bosonic mode below a threshold, i. e. $\gamma \lesssim 4$. As shown in the inset of Fig. 4(b) the bosonic squeezing parameter ξ_b takes its minimum values near $\lambda = \lambda_c(\gamma)$.

VI. DISCUSSION

We explored the stationary structure of correlations in an open quantum Dicke model. We found that, in the absence of local spin-decay ($\gamma = 0$), the superradiant phase does not feature spin-boson entanglement. Even though this may appear somehow counter-intuitive since superradiant phases arise in the strong coupling regime [67], disentangled nonequilibrium superradiant phases have also been observed in other settings [68]. However, as we have shown, the presence of local spin-decay ($\gamma \neq 0$) appears to be beneficial for the build-up of quantum correlations and can even be used to "stabilize" entanglement in superradiant stationary regimes. Furthermore, through other measures of correlations, we have shown that, even when there is no spin-boson entanglement, there are residual quantum correlations in the system which evidence non-classical properties across the whole phase diagram of the open quantum Dicke model. Also these correlations, and not only entanglement, could be exploited to achieve quantum-enhanced sensitivity in metrological applications [69].

While the model considered here can also be efficiently investigated numerically [24, 32], our analytical approach provides key insight on the behavior of fluctuations, which is necessary to quantify quantum correlations. For instance, from bare numerical simulations, it would not be possible to completely characterize entanglement, since the logarithmic negativity could be zero even for entangled states in generic systems, nor the quantum discord since it is not known how to compute this quantity for the finite-N (mixed) stationary state of a many-body open quantum system.

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APPENDIX A: STABILITY ANALYSIS OF THE STATIONARY MEAN-FIELD SOLUTIONS

In Sec. III we discussed the derivation of the mean-field equations, Eqs. (8), and have presented their stable stationary solutions. The purpose of this Appendix is to provide details on the stability analysis, which we perform using Lyapunov's indirect method [70].

For $\gamma = 0$ the constraint ${m^z}^2 = \frac{1}{4} - {m^y}^2 - {m^x}^2$, occurring due to the conservation of $\vec{S}^2$, reduces Eqs. (8) to a system of four coupled first-order non-linear differential equations

$$\begin{aligned} \dot{m^x} &= -\omega_z m^y \\ \dot{m^y} &= \omega_z m^x \mp 2^{3/2}\lambda m^p \sqrt{\frac{1}{4} - {m^y}^2 - {m^x}^2} \\ \dot{m^q} &= \omega_m m^p + 2^{3/2}\lambda m^x - \frac{\kappa}{2} m^q \\ \dot{m^p} &= -\omega_m m^q - \frac{\kappa}{2} m^p \end{aligned}$$

which completely determines the dynamics. It can be written in the form $\dot{\vec{u}} = f(\vec{u})$ with $\vec{u} = (m^x, m^y, m^q, m^p)^T$ and the Jacobian of f is

$$J(\vec{u}) = \begin{pmatrix} 0 & -\omega_z & 0 & 0 \\ \omega_z \pm 2^{3/2}\lambda m^p \frac{m^x}{\sqrt{\frac{1}{4} - {m^y}^2 - {m^x}^2}} & \pm 2^{3/2}\lambda m^p \frac{m^y}{\sqrt{\frac{1}{4} - {m^y}^2 - {m^x}^2}} & 0 & \mp 2^{3/2}\lambda \sqrt{\frac{1}{4} - {m^y}^2 - {m^x}^2} \\ 2^{3/2}\lambda & 0 & -\frac{\kappa}{2} & \omega_m \\ 0 & 0 & -\omega_m & -\frac{\kappa}{2} \end{pmatrix}.$$

The characteristic polynomial is calculated as

$$\det(\mathrm{J}(\vec{u}) - \mathbb{1}\chi) = a_0\chi^4 + a_1\chi^3 + a_2\chi^2 + a_3\chi + a_4$$

with coefficients

$$\begin{aligned} a_0 =& 1 \\ a_1 =& \kappa + \frac{2^{3/2} m^y m^p \lambda}{\sqrt{\frac{1}{4} - {m^x}^2 - {m^y}^2}} \\ a_2 =& \frac{\kappa^2}{4} + \frac{2^{3/2} m^y m^p \kappa\lambda}{\sqrt{\frac{1}{4} - {m^x}^2 - {m^y}^2}} + \omega_m^2 - \frac{2^{3/2} m^x m^p \lambda\omega_z}{\sqrt{\frac{1}{4} - {m^x}^2 - {m^y}^2}} + \omega_z^2 \\ a_3 =& \frac{m^y m^p \kappa^2 \lambda}{\sqrt{2}\sqrt{\frac{1}{4} - {m^x}^2 - {m^y}^2}} + \frac{2^{3/2} m^y m^p \lambda\omega_m^2}{\sqrt{\frac{1}{4} - {m^x}^2 - {m^y}^2}} - \frac{2^{3/2} m^x m^p \kappa\lambda\omega_z}{\sqrt{\frac{1}{4} - {m^x}^2 - {m^y}^2}} + \kappa\omega_z^2 \\ a_4 =& -\frac{m^x m^p \kappa^2 \lambda\omega_z}{\sqrt{2}\sqrt{\frac{1}{4} - {m^x}^2 - {m^y}^2}} - 8\sqrt{\frac{1}{4} - {m^x}^2 - {m^y}^2}\,\lambda^2\omega_m\omega_z - \frac{2^{3/2} m^x m^p \lambda\omega_m^2\omega_z}{\sqrt{\frac{1}{4} - {m^x}^2 - {m^y}^2}} + \frac{\kappa^2\omega_z^2}{4}. \end{aligned}$$

Then, Hurwitz' theorem [71, 72] states that all roots of this polynomial have negative real parts (which is a sufficient condition for stability of the solution) if and only if the inequalities

$$\begin{aligned} a_1 &> 0 \\ a_1 a_2 - a_0 a_3 &> 0 \\ (a_1 a_2 - a_0 a_3) a_3 - a_1^2 a_4 &> 0 \\ a_4 &> 0 \end{aligned}$$

hold. Employing this theorem we see that for our choice of parameters $\omega_z = 4$, $\omega_m = 1$, $\kappa = 1$ all roots of the characteristic polynomial have a negative real part if and only if, for $0 < \lambda < \lambda_c$, the stationary solution is the one in Eq. (11), or, for $\lambda > \lambda_c$, the stationary solution is either of the two in Eqs. (12). Then the respective stationary solutions are asymptotically stable [70]. Moreover, numerical evidence shows that for $\lambda = \lambda_c$ small perturbations of the solution in Eq. (11) (coinciding at this point with the solutions in Eqs. (12)) as initial conditions of the dynamics still drive the system to a stationary state where $m^z = -1/2,\ m^x = m^y = m^q = m^p = 0$.
We proceed with the case $\gamma > 0$. Here the Jacobian is

$$J(\vec{m}) = \begin{pmatrix} -\frac{\gamma}{2} & -\omega_z & 0 & 0 & 0 \\ \omega_z & -\frac{\gamma}{2} & -2^{3/2}\lambda m^p & 0 & -2^{3/2}\lambda m^z \\ 0 & 2^{3/2}\lambda m^p & -\gamma & 0 & 2^{3/2}\lambda m^y \\ 2^{3/2}\lambda & 0 & 0 & -\frac{\kappa}{2} & \omega_m \\ 0 & 0 & 0 & -\omega_m & -\frac{\kappa}{2} \end{pmatrix}$$

and the coefficients of the characteristic polynomial

$$\det(\mathrm{J}(\vec{m}) - \mathbb{1}\chi) = -b_0\chi^5 - b_1\chi^4 - b_2\chi^3 - b_3\chi^2 - b_4\chi - b_5$$

are given by

$$\begin{aligned}
b_0 =& 1 \\
b_1 =& 2\gamma + \kappa \\
b_2 =& \frac{5}{4}\gamma^2 + 2\gamma\kappa + \frac{1}{4}\kappa^2 + 8m^{p2}\lambda^2 + \omega_m^2 + \omega_z^2 \\
b_3 =& \frac{1}{4}\gamma^3 + \frac{5}{4}\gamma^2\kappa + \frac{1}{2}\gamma\kappa^2 + 4m^{p2}\gamma\lambda^2 + 8m^{p2}\kappa\lambda^2 + 2\gamma\omega_m^2 + \gamma\omega_z^2 + \kappa\omega_z^2 \\
b_4 =& \frac{1}{4}\gamma^3\kappa + \frac{5}{16}\gamma^2\kappa^2 + 4m^{p2}\gamma\kappa\lambda^2 + 2m^{p2}\kappa^2\lambda^2 + \frac{5}{4}\gamma^2\omega_m^2 + 8m^{p2}\lambda^2\omega_m^2 + 8m^z\lambda^2\omega_m\omega_z + \gamma\kappa\omega_z^2 + \frac{1}{4}\kappa^2\omega_z^2 + \omega_m^2\omega_z^2 \\
b_5 =& \frac{1}{16}\gamma^3\kappa^2 + m^{p2}\gamma\kappa^2\lambda^2 + \frac{1}{4}\gamma^3\omega_m^2 + 4m^{p2}\gamma\lambda^2\omega_m^2 + 8m^z\gamma\lambda^2\omega_m\omega_z + 2^{9/2}m^y m^p\lambda^3\omega_m\omega_z + \frac{1}{4}\gamma\kappa^2\omega_z^2 + \gamma\omega_m^2\omega_z^2 \,.
\end{aligned}$$

All roots of this degree 5 polynomial have negative real parts [71, 72] if and only if the inequalities

$$\begin{aligned}
b_1 &> 0 \\
b_1b_2 - b_0b_3 &> 0 \\
(b_1b_2 - b_0b_3)b_3 - b_1^2b_4 + b_0b_1b_5 &> 0 \\
((b_1b_2 - b_0b_3)b_3 - b_1^2b_4 + b_0b_1b_5)b_4 + (b_2b_3 + b_1b_4)b_0b_5 - b_0^2b_5^2 - b_1b_2^2b_5 &> 0 \\
b_5 &> 0
\end{aligned}$$

hold. For $\omega_z = 4$, $\omega_m = 1$, $\kappa = 1$ the solution in Eq. (11) is asymptotically stable for $0 \le \lambda < \lambda_c$. On the other hand, for $\lambda > \lambda_c$ the solutions in Eqs. (10) are stable. Also in this case there is numerical evidence that for $\lambda = \lambda_c$ the (unique) stationary solution is approached eventually.

APPENDIX B: TIME-EVOLUTION OF THE COVARIANCE MATRIX

In this Appendix we complete our discussion of Sec. IV, giving the full calculation yielding the form of the (parameter) matrices $\tilde{G}$ and $\tilde{W}$. Starting from Eqs. (16) we consider first the commutator of the Dicke Hamiltonian with the fluctuation vector components for which

$$i[H_N^D, F_N^\alpha] = \sum_{\eta\in\{x,y,z\}} \delta^{\alpha\eta}\Big(\frac{-\omega_z}{\sqrt{N}}\epsilon^{z\alpha\gamma}S^\gamma - \frac{2^{3/2}}{\sqrt{N}}\lambda m_N^p\epsilon^{x\alpha\gamma}S^\gamma\Big) + \delta^{\alpha q}\Big(\omega_m p + \frac{2^{3/2}\lambda}{\sqrt{N}}S^x\Big) - \delta^{\alpha p}\omega_m q.$$

Exploiting the fact that $\langle F_N^\alpha\rangle = 0$, we can write

$$\begin{aligned}\langle i[H_N^D, F_N^\alpha]F_N^\beta\rangle = \langle i[H_N^D, F_N^\alpha]F_N^\beta\rangle - \langle i[H_N^D, F_N^\alpha]\rangle\langle F_N^\beta\rangle \overset{N\gg 1}{=} &\sum_{\eta\in\{x,y,z\}}\delta^{\alpha\eta}(-\omega_z\epsilon^{z\alpha\gamma}\langle F^\gamma F^\beta\rangle - 2^{3/2}\lambda\epsilon^{x\alpha\gamma}m^\gamma\langle F^p F^\beta\rangle \\ &- 2^{3/2}\lambda\epsilon^{x\alpha\gamma}m^p\langle F^\gamma F^\beta\rangle) + \delta^{\alpha q}(\omega_m\langle F^p F^\beta\rangle \\ &+ 2^{3/2}\lambda\langle F^x F^\beta\rangle) - \delta^{\alpha p}\omega_m\langle F^q F^\beta\rangle \\ &=: A^{\alpha\gamma}C^{\gamma\beta} = (AC)^{\alpha\beta}\end{aligned}$$

with

$$A = \begin{pmatrix} 0 & -\omega_z & 0 & 0 & 0 \\ \omega_z & 0 & -2^{3/2}\lambda m^p & 0 & -2^{3/2}\lambda m^z \\ 0 & 2^{3/2}\lambda m^p & 0 & 0 & 2^{3/2}\lambda m^y \\ 2^{3/2}\lambda & 0 & 0 & 0 & \omega_m \\ 0 & 0 & 0 & -\omega_m & 0 \end{pmatrix}.$$

In the above calculation we have used that $m_N^\alpha \to m^\alpha$, multiple of the identity, in the thermodynamic limit.

Analogously, we can calculate

$$\begin{aligned}\langle iF_N^\alpha[H_N^D, F_N^\beta]\rangle \overset{N\gg 1}{=} &\sum_{\eta\in\{x,y,z\}}\delta^{\beta\eta}(-\omega_z\epsilon^{z\beta\gamma}\langle F^\alpha F^\gamma\rangle - 2^{3/2}\lambda\epsilon^{x\beta\gamma}m^\gamma\langle F^\alpha F^p\rangle - 2^{3/2}\lambda\epsilon^{x\beta\gamma}m^p\langle F^\alpha F^\gamma\rangle) \\ &+ \delta^{\beta q}(\omega_m\langle F^\alpha F^p\rangle + 2^{3/2}\lambda\langle F^\alpha F^x\rangle) - \delta^{\beta p}\omega_m\langle F^\alpha F^q\rangle \\ &=: C^{\alpha\gamma}B^{\gamma\beta} = (CB)^{\alpha\beta}\end{aligned}$$

with

$$B = \begin{pmatrix} 0 & \omega_z & 0 & 2^{3/2}\lambda & 0 \\ -\omega_z & 0 & 2^{3/2}\lambda m^p & 0 & 0 \\ 0 & -2^{3/2}\lambda m^p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega_m \\ 0 & -2^{3/2}\lambda m^z & 2^{3/2}\lambda m^y & \omega_m & 0 \end{pmatrix} = A^T.$$

For the dissipative contributions in $\dot{C}_N$ we expand them as

$$\frac{1}{2}([A^\dagger, F_N^\alpha F_N^\beta]A + A^\dagger[F_N^\alpha F_N^\beta, A]) = F_N^\alpha(\frac{1}{2}([A^\dagger, F_N^\beta]A + A^\dagger[F_N^\beta, A])) + (\frac{1}{2}([A^\dagger, F_N^\alpha]A + A^\dagger[F_N^\alpha, A]))F_N^\beta + [A^\dagger, F_N^\alpha][F_N^\beta, A]$$

to achieve

$$\mathcal{D}_N^\kappa[F_N^\alpha F_N^\beta] = F_N^\alpha\mathcal{D}_N^\kappa[F_N^\beta] + \mathcal{D}_N^\kappa[F_N^\alpha]F_N^\beta + \frac{N\kappa}{2}[m_N^p + im_N^q, F_N^\alpha][F_N^\beta, m_N^p - im_N^q]$$

and

$$\mathcal{D}_N^\gamma[F_N^\alpha F_N^\beta] = F_N^\alpha\mathcal{D}_N^\gamma[F_N^\beta] + \mathcal{D}_N^\gamma[F_N^\alpha]F_N^\beta + \gamma\sum_{k=1}^N[\sigma_k^+, F_N^\alpha][F_N^\beta, \sigma_k^-]. \tag{B1}$$

Focusing on $\mathcal{D}_N^\kappa$, we find for the last term on the right-hand side

$$\frac{\kappa}{2}[p + iq, F_N^\alpha][F_N^\beta, p - iq] = \frac{\kappa}{2}(-s_N^{\alpha p}s_N^{p\beta} - is_N^{\alpha q}s_N^{p\beta} + is_N^{\alpha p}s_N^{q\beta} - s_N^{\alpha q}s_N^{q\beta}) =: -s_N^{\alpha\gamma}D'^{\gamma\delta}s_N^{\delta\beta} = (-s_N D' s_N)^{\alpha\beta}$$

where the *symplectic matrix* s_N is given by the commutation relations of the fluctuation operators $s_N^{\alpha\beta} = -i[F_N^\alpha, F_N^\beta]$ and explicitly

$$s_N \overset{N\gg 1}{=} \begin{pmatrix} 0 & m^z & -m^y & 0 & 0 \\ -m^z & 0 & m^x & 0 & 0 \\ m^y & -m^x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} = s.$$

Furthermore

$$D' = \frac{\kappa}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & -i & 1 \end{pmatrix}.$$

The single-fluctuation κ-dissipator reads

$$\mathcal{D}_N^\kappa[F_N^\alpha] = \frac{\kappa}{4}(is_N^{p\alpha}p + s_N^{p\alpha}q - s_N^{q\alpha}p + is_N^{q\alpha}q + ips_N^{\alpha p} - qs_N^{\alpha p} + ps_N^{\alpha q} + iqs_N^{\alpha q})$$

and therefore

$$\langle \mathcal{D}_N^\kappa[F_N^\alpha]F_N^\beta\rangle \overset{N\gg 1}{=} \frac{\kappa}{2}(s^{\alpha q}\langle F^pF^\beta\rangle - s^{\alpha p}\langle F^qF^\beta\rangle) =: s^{\alpha\gamma}E^{\gamma\delta}C^{\delta\beta} = (sEC)^{\alpha\beta}.$$

Analogously

$$\langle F_N^\alpha \mathcal{D}_N^\kappa[F_N^\beta]\rangle \overset{N\gg 1}{=} \frac{\kappa}{2}(-\langle F^\alpha F^p\rangle s^{q\beta} + \langle F^\alpha F^q\rangle s^{p\beta}) =: C^{\alpha\gamma}E'^{\gamma\delta}s^{\delta\beta} = (CE's)^{\alpha\beta}$$

with

$$E = \frac{\kappa}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} = E'.$$

Collecting intermediately all the results concerning $\mathcal{D}_N^\kappa$, we see

$$\langle \mathcal{D}_N^\kappa[F_N^\alpha F_N^\beta]\rangle \overset{N\gg 1}{=} (CEs + sEC - sD's)^{\alpha\beta}.$$

Proceeding with $\mathcal{D}_N^\gamma$, we have for the single-fluctuation dissipator

$$\begin{aligned}\mathcal{D}_N^\gamma[F_N^\alpha] =& \frac{\gamma}{2\sqrt{N}}\sum_{k=1}^N(\delta^{x\alpha}(\sigma_k^z\sigma_k^- + \sigma_k^+\sigma_k^z) + \delta^{y\alpha}(i\sigma_k^z\sigma_k^- - i\sigma_k^+\sigma_k^z) + \delta^{z\alpha}(-\sigma_k^+\sigma_k^- - \sigma_k^+\sigma_k^-)) \\ =& \frac{\gamma}{2}(\delta^{x\alpha}(-\frac{S^x}{\sqrt{N}}) + \delta^{y\alpha}(-\frac{S^y}{\sqrt{N}}) + \delta^{z\alpha}(-\sqrt{N}\mathbb{1} - 2\frac{S^z}{\sqrt{N}}))\end{aligned}$$

and

$$\langle \mathcal{D}_N^\gamma[F_N^\alpha]F_N^\beta\rangle \overset{N\gg 1}{=} \frac{\gamma}{2}(-\delta^{x\alpha}\langle F^xF^\beta\rangle - \delta^{y\alpha}\langle F^yF^\beta\rangle - \delta^{z\alpha}2\langle F^zF^\beta\rangle) =: Q^{\alpha\gamma}C^{\gamma\beta} = (QC)^{\alpha\beta}$$

with

$$Q = \begin{pmatrix} -\frac{\gamma}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{\gamma}{2} & 0 & 0 & 0 \\ 0 & 0 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Also, we have that

$$\langle F_N^\alpha \mathcal{D}_N^\gamma[F_N^\beta]\rangle \overset{N\gg 1}{=} \frac{\gamma}{2}(-\delta^{x\beta}\langle F^\alpha F^x\rangle - \delta^{y\beta}\langle F^\alpha F^y\rangle - \delta^{z\beta}2\langle F^\alpha F^z\rangle) = C^{\alpha\gamma}Q^{\gamma\beta} = (CQ)^{\alpha\beta}.$$

For the last term in Eq. (B1) we get

$$\begin{aligned}\gamma\sum_{k=1}^N[\sigma_k^+, F_N^\alpha][F_N^\beta, \sigma_k^-] =& \frac{\gamma}{N}\sum_{k=1}^N(\delta^{x\alpha}\delta^{x\beta}\frac{\mathbb{1}_k}{4} + \delta^{x\alpha}\delta^{y\beta}\frac{-i\mathbb{1}_k}{4} + \delta^{x\alpha}\delta^{z\beta}(-\sigma_k^z\sigma_k^-) \\ &+ \delta^{y\alpha}\delta^{x\beta}\frac{i\mathbb{1}_k}{4} + \delta^{y\alpha}\delta^{y\beta}\frac{\mathbb{1}_k}{4} + \delta^{y\alpha}\delta^{z\beta}(-i\sigma_k^z\sigma_k^-) \\ &+ \delta^{z\alpha}\delta^{x\beta}(-\sigma_k^+\sigma_k^z) + \delta^{z\alpha}\delta^{y\beta}i\sigma_k^+\sigma_k^z + \delta^{z\alpha}\delta^{z\beta}\sigma_k^+\sigma_k^-)\end{aligned}$$

and by means of $\sigma_k^\rho \sigma_k^\nu = \delta^{\rho\nu}\frac{\mathbb{1}_k}{4} + i\epsilon^{\rho\nu\mu}\frac{\sigma_k^\mu}{2}$,

$$\begin{aligned}\langle \gamma \sum_{k=1}^{N} [\sigma_k^+, F_N^\alpha][F_N^\beta, \sigma_k^-]\rangle \overset{N\gg 1}{=} \gamma(&\delta^{x\alpha}\delta^{x\beta}\frac{1}{4} + \delta^{x\alpha}\delta^{y\beta}\frac{-i}{4} + \delta^{x\alpha}\delta^{z\beta}\frac{m^x - im^y}{2} \\ &+ \delta^{y\alpha}\delta^{x\beta}\frac{i}{4} + \delta^{y\alpha}\delta^{y\beta}\frac{1}{4} + \delta^{y\alpha}\delta^{z\beta}\frac{m^y + im^x}{2} \\ &+ \delta^{z\alpha}\delta^{x\beta}\frac{m^x + im^y}{2} + \delta^{z\alpha}\delta^{y\beta}\frac{m^y - im^x}{2} \\ &+ \delta^{z\alpha}\delta^{z\beta}(\frac{1}{2} + m^z)) =: Z'^{\alpha\beta}.\end{aligned}$$

Here

$$Z' = \gamma \begin{pmatrix} \frac{1}{4} & \frac{-i}{4} & \frac{m^x - im^y}{2} & 0 & 0 \\ \frac{i}{4} & \frac{1}{4} & \frac{m^y + im^x}{2} & 0 & 0 \\ \frac{m^x + im^y}{2} & \frac{m^y - im^x}{2} & \frac{1}{2} + m^z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, collecting the results concerning $\mathcal{D}_N^\gamma$ gives

$$\langle \mathcal{D}_N^\gamma [F_N^\alpha F_N^\beta]\rangle \overset{N\gg 1}{=} (CQ + QC + Z')^{\alpha\beta}$$

and we conclude for Eq. (16) in the thermodynamic limit

$$\dot{C} = CA^T + AC + CEs + sEC - sD's + CQ + QC + Z'.$$

Considering then Eq. (15), we finally get the differential equation for the covariance matrix

$$\begin{aligned}\dot{\Sigma} =& \Sigma(A^T + Es + Q) + (A + sE + Q)\Sigma - \frac{sD's + s^T D'^T s^T}{2} + \frac{Z' + Z'^T}{2} \\ =& \Sigma G^T + G\Sigma - sDs + Z\end{aligned}$$

where we defined $G := A + sE + Q$ and

$$Z := \gamma \begin{pmatrix} \frac{1}{4} & 0 & \frac{m^x}{2} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{m^y}{2} & 0 & 0 \\ \frac{m^x}{2} & \frac{m^y}{2} & \frac{1}{2} + m^z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad D := \frac{\kappa}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We note that the differential equation for the covariance matrix involves, in general, time-dependent matrices G, s and Z. These may indeed be time-dependent through the time-dependence of the mean-field operators which appear in their matrix elements. However, as stated in the main text, for the purpose of this work the matrices G, s and Z are time-independent since we investigate here the behavior of fluctuations when the initial state of the system is stationary with respect to the mean-field observables. As a consequence, this differential equation for the covariance matrix in the original frame can be connected to the one in Eq. (17) by considering Eq. (14) and that

$$\tilde{G} = JR_{\hat{n}} G R_{\hat{n}}^T J^{-1}, \qquad \tilde{W} = JR_{\hat{n}}(Z - sDs)R_{\hat{n}}^T J.$$

We want to conclude this section by elaborating on the fact that local decay as a dissipative process acting on fluctuation operators does not spoil the Gaussian character of the latter (a full proof of this statement can be found in Ref. [40]). Since the decay only acts non-trivially on spin operators, we restrict the analysis here to spin fluctuation operators.

We simply consider the time-evolution generated by the dissipator $\mathcal{D}_N^\gamma$ and study the emergent characteristic function (i.e. the Fourier transform of the probability distribution) of a quantum fluctuation operator F_N^α. This is defined as follows

$$\chi(\ell) = \lim_{N\to\infty} \left\langle e^{i\ell F_N^\alpha} \right\rangle_t,$$

where, here, $\langle\cdot\rangle_t$ is the expectation value over a quantum state which has evolved up to time t solely through a dissipative dynamics involving local decay. We further assume, for the sake of simplicity, that the initial state is a product state, i.e., $\langle\sigma_k^\alpha\sigma_h^\beta\rangle = \langle\sigma_k^\alpha\rangle\langle\sigma_h^\beta\rangle$, for all $h \neq k$ and that it is further translation invariant $\langle\sigma_k^\alpha\rangle = \langle\sigma_h^\alpha\rangle$, for all h, k. Given that local decay is a process which occurs independently from site to site, the quantum state remains product and translation invariant at every time. This observation makes the computation of $\chi(\ell)$ rather straightforward.

Writing the exponential of the fluctuation operator as a product of single site exponentials, and exploiting the fact that the quantum state is a product state at any time, we have

$$\left\langle e^{i\ell F_N^\alpha}\right\rangle_t = \prod_{k=1}^N \left\langle e^{\frac{i\ell}{\sqrt{N}}(\sigma_k^\alpha - \langle\sigma_k^\alpha\rangle_t)}\right\rangle_t .$$

Furthermore, exploiting the translation invariance of the state and expanding the exponentials we find

$$\left\langle e^{i\ell F_N^\alpha}\right\rangle_t = \left[\left\langle 1 + \frac{i\ell}{\sqrt{N}}(\sigma^\alpha - \langle\sigma^\alpha\rangle_t) - \frac{\ell^2}{2N}(\sigma^\alpha - \langle\sigma^\alpha\rangle_t)^2\right\rangle_t + O(N^{-3/2})\right]^N .$$

The term proportional to $N^{-1/2}$ evaluates to zero under expectation. The terms $O(N^{-3/2})$ can be neglected in the large N limit. Thus, taking $N \to \infty$ in the above relation, we arrive at the following expression

$$\chi(\ell) = e^{-\frac{\ell^2}{2}\Sigma^{\alpha\alpha}} ,$$

where

$$\Sigma^{\alpha\alpha} = \langle(\sigma^\alpha - \langle\sigma^\alpha\rangle_t)^2\rangle_t = \lim_{N\to\infty}\frac{1}{2}\langle\{F_N^\alpha, F_N^\alpha\}\rangle_t .$$

The above argument thus exploits the fact that local decay acts independently from site to site to show that this cannot break the Gaussianity of an initial product Gaussian state. This argument can be straightforwardly generalized to account for the characteristic function of all quantum fluctuation operators. It can further be generalized to more complex initial Gaussian states following the more technical proof in Ref. [40].

APPENDIX C: STATIONARY COVARIANCE MATRIX

In this Appendix we show how the stationary covariance matrix can be obtained through a vectorization procedure. We will refer to an odd-dimensional square matrix as in "cross form" if its central row and central column consist only of zeros. An even-dimensional square matrix arising from an odd-dimensional one M by deleting the central row and central column is said here to be in "reduced form" and we denote it by M_{red}.
We start with the differential equation for the covariance matrix in Eq. (17)

$$\dot{\tilde{\Sigma}}(t) = \tilde{\Sigma}(t)\tilde{G}^T + \tilde{G}\tilde{\Sigma}(t) + \tilde{W}. \qquad \text{(C1)}$$

We focus on parameters chosen for the figures in the main text, i. e. $\omega_z = 4$, $\omega_m = 1$, $\kappa = 1$. Let $I \subseteq \mathbb{R}$ be an open interval. Here $t \in I$ and $t > t_0 \in I$. At t_0 it is assumed that the quantum state is such that

$$\vec{m}(t_0) = \begin{cases} \vec{m}_{\mathrm{sub}} & \text{for } \lambda \in [0, \lambda_c] \\ \vec{m}_{\mathrm{sup}} & \text{for } \lambda \in (\lambda_c, \infty) , \end{cases}$$

where $\vec{m}_{\mathrm{sub}}$ is the vector containing the stable solution of the mean-field equations in the normal phase, while $\vec{m}_{\mathrm{sup}}$ is the vector containing the solution of the mean-field equations in the superradiant phase.

The cases $\gamma = 0$ and $\gamma > 0$ are treated separately and we first focus on $\gamma > 0$. The task is to find the stationary covariance matrix $\tilde{\Sigma}_\infty$ which is such that $\dot{\tilde{\Sigma}}_\infty = 0$. The matrix equation to be solved, given by

$$\tilde{\Sigma}_\infty\tilde{G}^T + \tilde{G}\tilde{\Sigma}_\infty = -\tilde{W} ,$$

is equivalent [73] to finding the 25 unknowns of the following linear system of 25 equations

$$(\tilde{G} \otimes \mathbb{1}_5 + \mathbb{1}_5 \otimes \tilde{G}) vec(\tilde{\Sigma}_\infty) = vec(-\tilde{W}) \qquad \text{(C2)}$$

where $\otimes$ is the Kronecker product and the operation $vec(\cdot)$ acts on a generic $(m \times n)$-matrix $K = (K_{*1}, K_{*2}, ..., K_{*n})$ (with K_{*i} the i-th column vector) as

$$K \mapsto vec(K) = (K_{*1}^T, K_{*2}^T, ..., K_{*n}^T)^T .$$

Eq. (C2) has a unique solution if and only if $\tilde{G} \otimes \mathbb{1}_5 + \mathbb{1}_5 \otimes \tilde{G}$ is invertible. The solution is, in vectorized form, given by

$$vec(\tilde{\Sigma}_\infty) = (\tilde{G} \otimes \mathbb{1}_5 + \mathbb{1}_5 \otimes \tilde{G})^{-1} vec(-\tilde{W}).$$

Equivalently to invertibility, we want to prove that any eigenvalue of $\tilde{G} \otimes \mathbb{1}_5 + \mathbb{1}_5 \otimes \tilde{G}$ is non-zero. If the spectrum of $\tilde{G}$ is $\sigma(\tilde{G}) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ then the set of these eigenvalues is $\sigma(\tilde{G}\otimes\mathbb{1}_5+\mathbb{1}_5\otimes\tilde{G})=\{\mu_i+\mu_j|i=1,...,5,j=1,...,5\}$. Thus, any eigenvalue is nonzero if $\sigma(\tilde{G})\cap\sigma(-\tilde{G})=\emptyset$, i.e. if no element of $\sigma(-\tilde{G})$ can be obtained by a point reflection of an element of $\sigma(\tilde{G})$ at the origin of the complex plane. Using again Hurwitz' theorem it can be proven that for $\lambda\neq\lambda_c$ all eigenvalues of $\tilde{G}$ lie in the open left half-plane. Consequently the matrix $\tilde{G}\otimes\mathbb{1}_5+\mathbb{1}_5\otimes\tilde{G}$ is invertible if $\lambda\neq\lambda_c$.

In the $\gamma=0$ case one cannot proceed the same way. In this setting, we focus on initial covariance matrices $\tilde{\Sigma}(t_0)$ that are in cross form (see the definition at the beginning of this Appendix). The matrix $\tilde{Z}$ is the zero matrix and

$$\tilde{s}=\begin{pmatrix}0&1&0&0&0\\-1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&-1&0\end{pmatrix},\ \tilde{G}=\begin{pmatrix}0&-\frac{2^{3/2}m^p\lambda m^x+m^z\omega_z}{|\vec{m}|}&0&0&0\\ \frac{2^{3/2}m^p\lambda m^x+m^z\omega_z}{|\vec{m}|}&0&0&0&\frac{-\operatorname{sgn}(m^x)2^{3/2}\lambda m^z}{\sqrt{|\vec{m}|}}\\0&0&0&0&0\\ \frac{\operatorname{sgn}(m^x)2^{3/2}\lambda m^z}{\sqrt{|\vec{m}|}}&0&\frac{2^{3/2}\lambda m^x}{|\vec{m}|}&-\frac{\kappa}{2}&\omega_c\\0&0&0&-\omega_c&-\frac{\kappa}{2}\end{pmatrix}$$

and $\tilde{D}=D$. With $e^{\tilde{G}(t-t_0)}=\sum_{n=0}^{\infty}\frac{\tilde{G}^n}{n!}\cdot(t-t_0)^n$ it is

$$\begin{aligned}&e^{\tilde{G}(t-t_0)}\tilde{\Sigma}(t_0)e^{\tilde{G}^T(t-t_0)}\\&=\lim_{m,n\to\infty}\sum_{k=0}^{m}\sum_{l=0}^{n}\frac{(t-t_0)^k(t-t_0)^l}{k!l!}\tilde{G}^k\tilde{\Sigma}(t_0)(\tilde{G}^T)^l.\end{aligned}$$

This is in cross form since $\tilde{\Sigma}(t_0)$ is in this form and thus $\tilde{G}^k\tilde{\Sigma}(t_0)(\tilde{G}^T)^l$ is in cross form, for all $k,l\in\mathbb{N}_0$. Similarly, since $\tilde{s}\tilde{D}\tilde{s}$ is in cross form, also $e^{\tilde{G}(t-s)}\tilde{s}\tilde{D}\tilde{s}e^{\tilde{G}^T(t-s)}$ is. It follows that the unique solution in Eq. (C3) has cross form for all $t\in I$ with $t>t_0$. Therefore it remains to solve

$$\begin{aligned}\dot{\tilde{\Sigma}}_{\rm red}(t)=&\dot{\tilde{\Sigma}}^{\rm t-m}(t)\\=&-\tilde{s}_{\rm red}\tilde{D}_{\rm red}\tilde{s}_{\rm red}+\tilde{\Sigma}^{\rm t-m}(t)\tilde{G}^T_{\rm red}+\tilde{G}_{\rm red}\tilde{\Sigma}^{\rm t-m}(t).\end{aligned}$$

the differential equation (C1) reduces to

$$\dot{\tilde{\Sigma}}(t)=-\tilde{s}\tilde{D}\tilde{s}+\tilde{\Sigma}(t)\tilde{G}^T+\tilde{G}\tilde{\Sigma}(t).$$

Known as the differential Sylvester equation [74], it has the unique solution

$$\begin{aligned}\tilde{\Sigma}(t)=&e^{\tilde{G}(t-t_0)}\tilde{\Sigma}(t_0)e^{\tilde{G}^T(t-t_0)}\\&-\int_{t_0}^{t}e^{\tilde{G}(t-s)}\tilde{s}\tilde{D}\tilde{s}e^{\tilde{G}^T(t-s)}ds.\end{aligned}\tag{C3}$$

We note that

We want to find the stationary covariance matrix $\dot{\tilde{\Sigma}}^{\rm t-m}_{\infty}(t)=0$. Solving the matrix equation

$$\tilde{\Sigma}^{\rm t-m}_{\infty}\tilde{G}^T_{\rm red}+\tilde{G}_{\rm red}\tilde{\Sigma}^{\rm t-m}_{\infty}=\tilde{s}_{\rm red}\tilde{D}_{\rm red}\tilde{s}_{\rm red}$$

is equivalent to finding the 16 unknowns of the linear system of 16 equations

$$(\tilde{G}_{\rm red}\otimes\mathbb{1}_4+\mathbb{1}_4\otimes\tilde{G}_{\rm red})vec(\tilde{\Sigma}^{\rm t-m}_{\infty})=vec(\tilde{s}_{\rm red}\tilde{D}_{\rm red}\tilde{s}_{\rm red}).$$

With the same steps as above, we establish the invertibility of $\tilde{G}_{\rm red}\otimes\mathbb{1}_4+\mathbb{1}_4\otimes\tilde{G}_{\rm red}$ for $\lambda\notin\{0,\lambda_c\}$ such that in this regime the unique stationary CM is given by

$$vec(\tilde{\Sigma}^{\rm t-m}_{\infty})=(\tilde{G}_{\rm red}\otimes\mathbb{1}_4+\mathbb{1}_4\otimes\tilde{G}_{\rm red})^{-1}vec(\tilde{s}_{\rm red}\tilde{D}_{\rm red}\tilde{s}_{\rm red}).$$

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