

# A new creep law for crushable aggregates

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The authors have recently proposed a new equation for the one-dimensional (1D) normal compression line, which contains a parameter controlling the size effect on average strength. They showed that the equation held for a wide range of discrete-element modelling (DEM) simulations of crushable aggregates. This paper incorporates the time-dependence of particle strength. A new equation is proposed and examined using DEM of 1D creep. The simulations show that while the plots may seem linear on a plot of voids ratio against the logarithm of time in the traditional way, the new proposed law, which is linear when the voids ratio is also plotted on a logarithmic scale, is more appropriate. The simulations examine the influence of the size effect hardening law, the time dependence on strength and stress level. It is shown that the new equation holds for each case.

**KEYWORDS:** compressibility; creep; discrete-element modelling; particle crushing/crushability; sands

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## NOTATION

$b$	size effect on strength for a material
$C$	creep coefficient
$d$	particle diameter
$e$	voids ratio
$e_0$	initial voids ratio
$e_c$	voids ratio under an applied stress $\sigma_c$
$e_y$	voids ratio corresponding to yield stress on $\log e - \log \sigma$ plot
$F$	force
$m$	Weibull modulus
$n$	slow-crack growth exponent
$q$	octahedral shear stress in a particle
$q_0$	value of octahedral shear stress such that 37% of particles are stronger for a given particle size
$t$	time
$t_0$	time from which creep strains are measured
$t(\text{test})$	time taken to measure tensile strength $\sigma_{TS}$
$\varepsilon$	creep strain
$\lambda$	slope of normal compression line in $\ln e - \ln \sigma$ space
$\sigma$	stress
$\sigma_1$	major principal stress in a particle
$\sigma_2$	intermediate principal stress in a particle
$\sigma_3$	minor principal stress in a particle
$\sigma_{av}$	average particle strength
$\sigma_c$	applied stress at a voids ratio of $e_c$
$\sigma_s$	strength of the smallest particle
$\sigma_{s0}$	strength of the smallest particle at time $t_0$
$\sigma_{TS}$	tensile strength
$\sigma_y$	yield stress on $\log e - \log \sigma$ plot

## INTRODUCTION

McDowell (2003) proposed a theoretical explanation for observed creep behaviour, which will be briefly described again here. Granular materials creep under constant effective stress (Leung *et al.*, 1996; Lade & Liu, 1998), such that creep strain is usually reported to be proportional to  $\log$  time

$$\varepsilon = C \log t/t_0 \quad (1)$$

where  $t_0$  is the time from which creep strains are measured. It is known (Leung *et al.*, 1996; Lade & Liu, 1998) that creep of granular materials is accompanied by particle crushing. McDowell & Bolton (1998) proposed that a linear normal compression line in voids ratio– $\log$  stress space

$$e = e_c - \lambda \ln(\sigma/\sigma_c) \quad (2)$$

was consistent with equation (1) for a granular material subjected to creep at constant stress under one-dimensional (1D) conditions. The assumptions were that the current macroscopic stress should be proportional to the average strength of the smallest particles in the aggregate: these particles continue to crush under increasing stress levels, becoming statistically stronger and filling voids. According to equation (2), an aggregate should be in equilibrium with a voids ratio  $e_c$  under an applied stress  $\sigma_c$ , where  $\sigma_c$  is proportional to the average strength of the current smallest particles  $\sigma_s$ , so that

$$\sigma_c = k\sigma_s \quad (3)$$

where  $k$  is independent of particle size due to self-similarity across different orders of particle size. Substituting equation (3) into equation (2) gives

$$e = e_c - \lambda \ln(\sigma/k\sigma_s) \quad (4)$$

The law for time-dependent strength of ceramics is that, for a tensile test on a ceramic specimen, if the standard test used to measure the tensile strength  $\sigma_{TS}$  takes time  $t(\text{test})$ , then the stress that the sample will support safely for a time  $t$  is given by (Davidge, 1979; Ashby & Jones, 1986)

$$\left(\frac{\sigma}{\sigma_{TS}}\right)^n = \frac{t(\text{test})}{t} \quad (5)$$

where  $n$  is the slow-crack growth exponent. Data for  $n$  are very limited, but  $n = 10\text{--}20$  for oxides at room temperature. It is widely accepted that the failure of a spherical body under diametral compression is in fact a tensile failure (e.g. Jaeger, 1967). Hence, if  $\sigma_{s0}$  is the average particle strength that can be measured at time  $t = t_0$ , then the average strength  $\sigma_s$  after time  $t$ , according to equation (5) would be

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$$\sigma_s = \sigma_{s0}(t_0/t)^{1/n} \tag{6}$$

Substituting equation (6) into equation (4) gives

$$e = e_c - \lambda \ln(\sigma/k\sigma_{s0}) - \frac{\lambda}{n} \ln(t/t_0) \tag{7}$$

Hence the reduction in voids ratio  $\Delta e$  as a function of time after time  $t_0$  is simply

$$\Delta e = \frac{\lambda}{n} \ln(t/t_0) = \frac{2.3\lambda}{n} \log(t/t_0) \tag{8}$$

so that the log time effect is observed. Taking an initial voids ratio  $e_0$  of 0.5, say, at the onset of creep, the creep coefficient in equation (1), given by

$$C = \frac{2.3\lambda}{n} \frac{1}{1+e_0} \tag{9}$$

ranges typically from about 0.0015 (taking  $\lambda = 0.1$ ,  $n = 100$ ) to 0.03 (taking  $\lambda = 0.2$ ,  $n = 10$ ). Most of the values of creep coefficients reported by Leung *et al.* (1996) for 1D compression of sand at high stress levels fall within this range.

A NEW CREEP LAW FROM A NEW NORMAL COMPRESSION LINE

McDowell & de Bono (2013) have shown, analytically and using discrete-element modelling (DEM), that the 1D normal compression law is actually given by

$$\log e = \log e_y - \frac{1}{2b} \log\left(\frac{\sigma}{\sigma_y}\right) \tag{10}$$

where  $b$  controls the particle size effect on average particle strength  $\sigma_{av}$

$$\sigma_{av} \propto d^{-b} \tag{11}$$

$e_y$  is the value on a linear log-log plot at a stress corresponding to the yield stress  $\sigma_y$  and  $\sigma_y$  is proportional to the average particle strength for an initially uniformly graded aggregate. In this case, if the analysis in the previous section is reapplied to the new McDowell & de Bono (2013) normal compression line, then

$$\log e = \log e_0 - \frac{1}{2bn} \log\left(\frac{t}{t_0}\right) \tag{12}$$

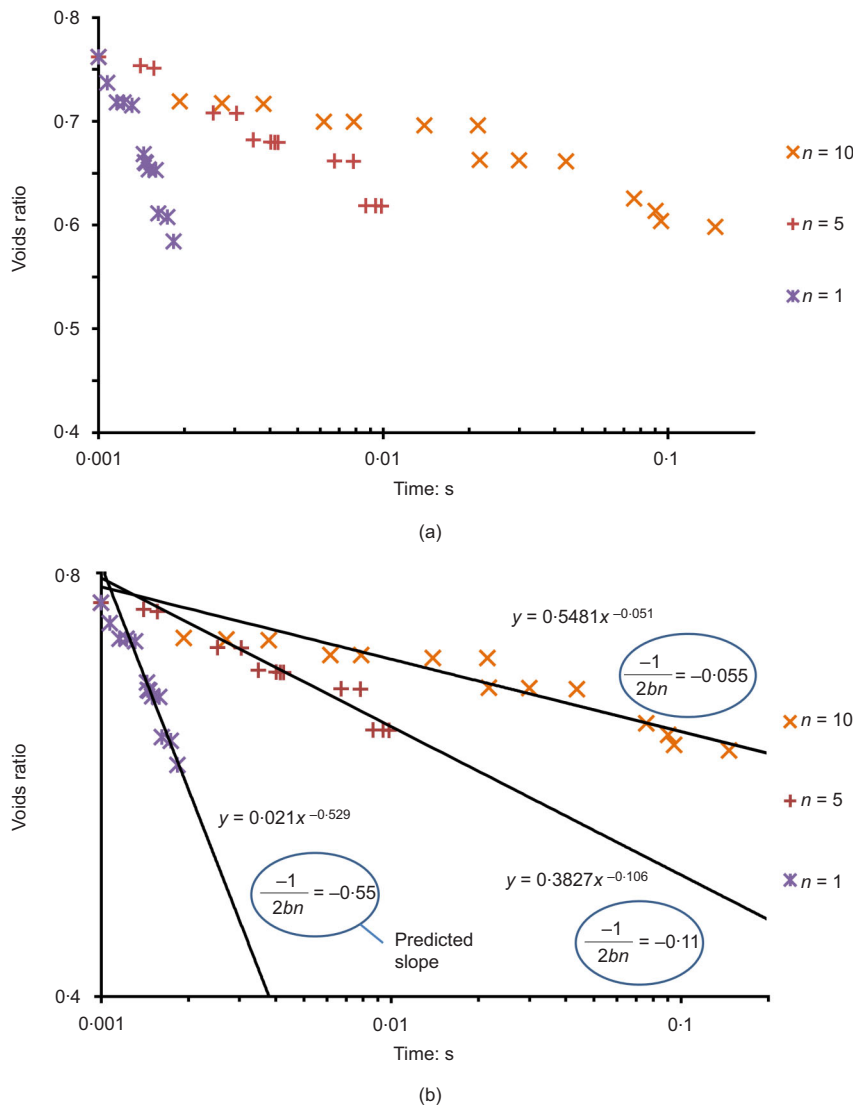


Fig. 1. Voids ratio as function of time for various time exponents  $n$ , plotted on conventional semi-logarithmic axes (a) and double logarithmic axes (b)

which is the new creep law for crushable granular materials proposed here. This paper uses the simple crushing model proposed by McDowell & de Bono (2013) to establish whether equation (12) applies to a simple sample of crushable particles subjected to normal compression and subsequently creep, using DEM.

DEM OF CREEP OF CRUSHABLE AGGREGATES

A dense random sample of 620 spheres of diameter 2 mm was created in a scaled-down oedometer of diameter 30 mm and height 7 mm and the sample was loaded one-dimensionally, in the same manner as described by McDowell & de Bono (2013). Using a larger oedometer is impractical due to the very large numbers of particles generated progressively throughout the simulations. To determine whether fracture should occur or not, the octahedral stress within each particle was used. This is given by

$$q = \frac{1}{3}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2]^{1/2} \tag{13}$$

This provides a simple criterion to facilitate breakage, taking into account multiple contacts on a particle surface

while avoiding the use of agglomerates. For PFC3D (Itasca, 2005), for a sphere compressed diametrically between two walls, the value of  $q$  generated using equation (13) is equivalent to

$$q = 0.9 \frac{F}{d^2} \tag{14}$$

and so is 0.9 times the induced characteristic stress. Therefore, the assumption was made that for particles loaded under multiple contacts, the particle would break if the octahedral shear stress was greater than or equal to its 'strength'. In this paper, the strengths of the particles satisfy a Weibull distribution of  $q$  values and, according to McDowell & de Bono (2013), if the Weibull modulus is  $m$  and the failure is governed by bulk fracture, then

$$b = \frac{3}{m} \tag{15}$$

Particle strength was also assumed to follow the time-dependent law given in equation (6). When a particle was found to have an induced value of  $q$  greater than or equal to its strength, it was replaced by two equal-sized fragments within the parent sphere, which overlapped so that the

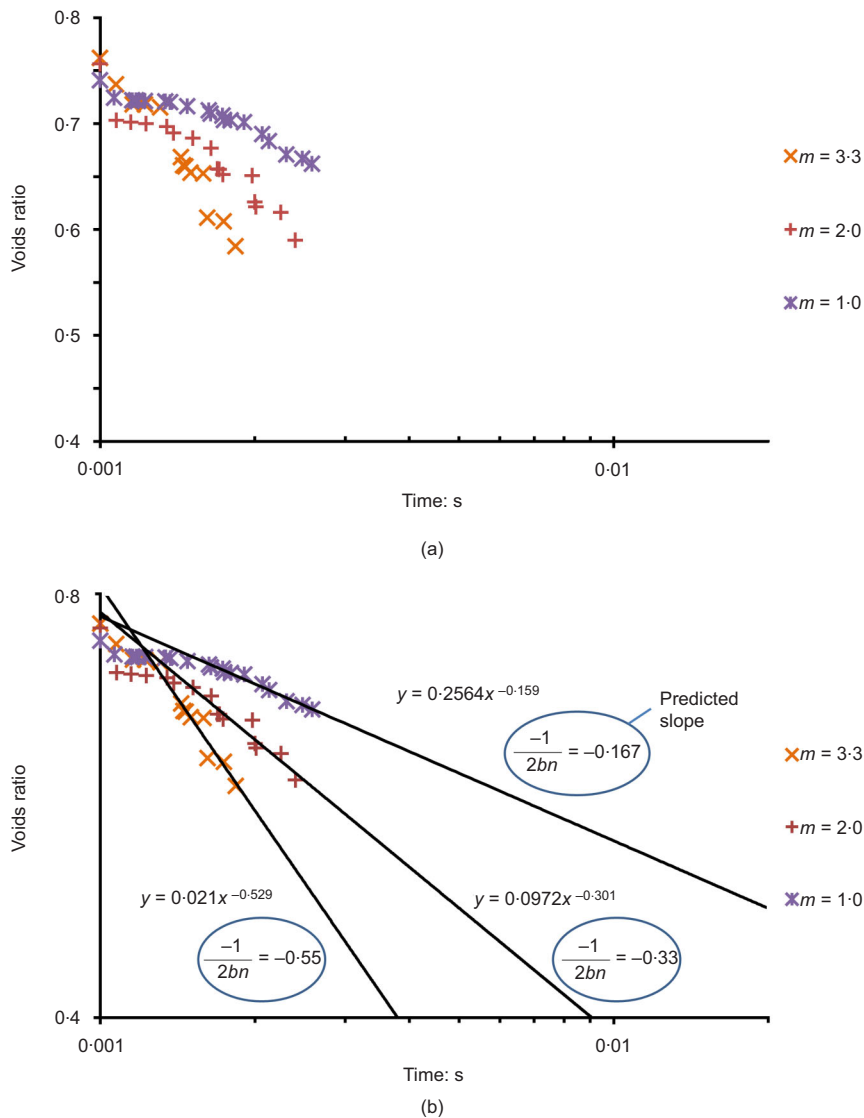


Fig. 2. Voids ratio as function of time for simulations with various values of Weibull modulus  $m$ , plotted on conventional semi-logarithmic axes (a) and double logarithmic axes (b)

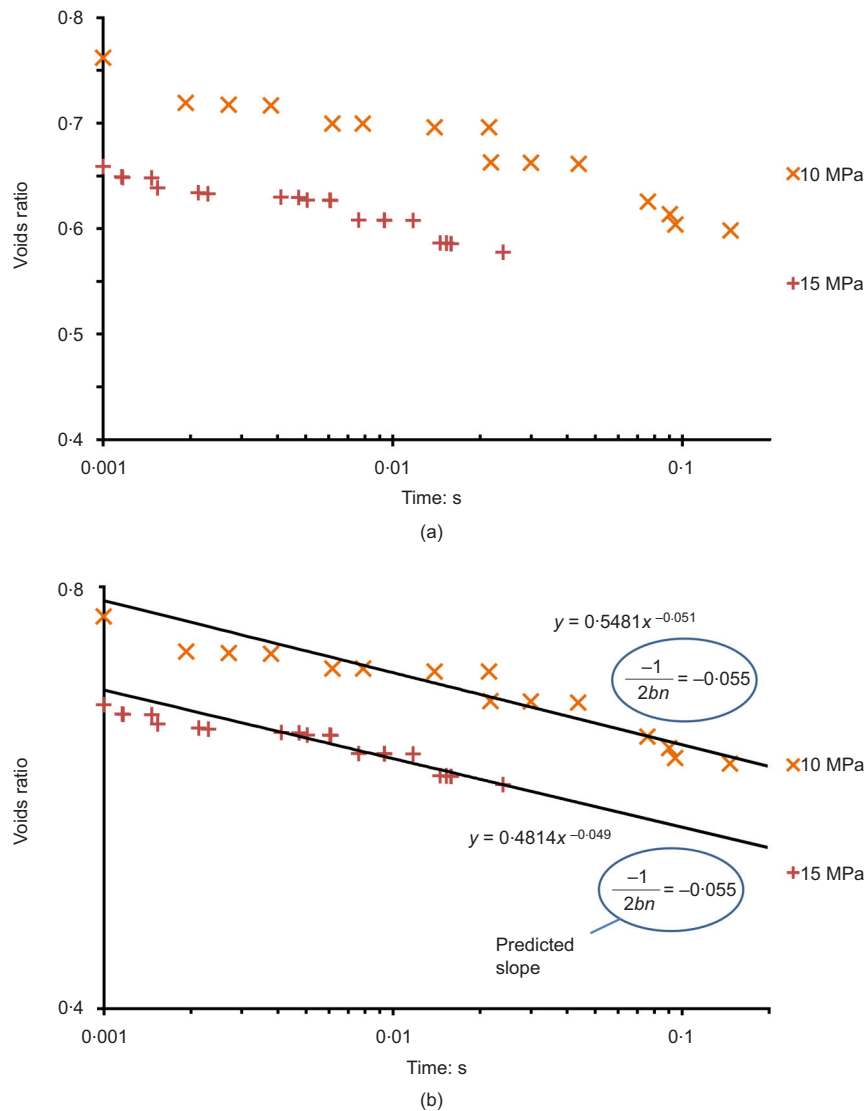
particles moved in the direction of the minor principal stress – as described by McDowell & de Bono (2013). The total volume of two fragments was equal to that of the parent sphere to ensure conservation of mass.

To simulate creep, a strength–time dependency was introduced to the model, as used and described in detail by McDowell & de Bono (2013). The oedometer sample was compressed beyond yield until the applied axial stress was 10 MPa for a sample of spheres having a Weibull modulus  $m$  of 3.3 and a 37%  $q_0$  strength of 37.5 MPa, which were the parameters previously used to model experimental data (McDowell, 2002). Then, using a value of  $t_0 = 0.001$  s, particle strengths were decreased according to equation (6). The simulation was cycled while continuously monitoring the octahedral shear stresses within each particle. When the stress within a particle was found to exceed the individual strength, it was replaced with two new fragments. Immediately afterwards, a number of computational cycles were required to dissipate the artificial pressure increase due to the overlap between new fragments, the same method as used by McDowell & de Bono (2013) to model 1D compression. During this period time was not considered. The top platen was then gradually reloaded to maintain a constant axial stress of 10 MPa. The stresses

were checked again and if any particles were found to be in a state of breakage, this process was repeated; that is, particles were replaced with new fragments, energy was allowed to dissipate and then the platen was reloaded to a stress of 10 MPa. Once no more particles were in a state of breakage under a constant axial stress of 10 MPa, the current voids ratio and elapsed time were recorded, after which consideration of time was resumed, and the strength–time dependency was applied.

Figure 1(a) shows the results for voids ratio as a function of log time for three different time exponents  $n$ . Figure 1(b) shows the same data plotted on a log–log scale according to equation (12). The slopes are shown and the predicted values according to equation (12) are also shown. If a larger value of  $t_0$  is used, any plots of voids ratio against log time would have the same slope according to equation (12), with the curve simply shifted to the right due to the higher starting value. However, larger increments of time would necessitate a higher number of computational timesteps to complete the simulations.

Figure 2 shows results for the same initial sample, with a time exponent  $n = 1$  and three different Weibull modulus values of 1.0, 2.0 and 3.3 (all with  $q_0 = 37.5$  MPa). Figure 2(a) shows the conventional plot of voids ratio



**Fig. 3.** Voids ratio as a function of time for simulations with the same strength characteristics and time exponent under effective stresses of 10 and 15 MPa, plotted on conventional semi-logarithmic axes (a) and double logarithmic axes (b)

against log time and Fig. 2(b) shows the results on the log–log plot, with the calculated slopes and the predicted values according to equation (12).

Figure 3(a) shows the creep response for the sample with  $q_0 = 37.5$  MPa,  $m = 3.3$  and  $n = 10$ , one-dimensionally loaded to stresses of 10 and 15 MPa. The results are plotted in the log–log space in Fig. 3(b). The simulation loaded to 15 MPa underwent a higher degree of compression, so has a lower voids ratio at the start of creep. Although the initial voids ratios are different, the slopes are the same and agree with the predicted value according to equation (12). The data points on the log  $e$ –log  $\sigma$  plots in all three figures do not fall on perfectly straight lines; this is simply because the initial sample contained only 620 spheres. However, the figures show clearly that equation (12) holds for each time exponent, size effect on strength and stress level considered.

## CONCLUSIONS

DEM has been used to show that the time-dependent law for the strength of ceramics gives rise to the correct creep behaviour under 1D conditions. The results agree with the hypothesis that the creep should be linear when the logarithm of voids ratio is plotted against the logarithm of time. The slope of the line has been shown to be given by a new equation (equation (12)), which also includes the size effect on average strength as well as the exponent for the time-dependent strength. Therefore, by performing standard tests to obtain the size effect on average tensile strength of grains of a material by crushing between flat

platens, and if the exponent for time-dependent strength can be measured by allowing particles to be loaded under constant stress and measuring the time to failure, then the creep behaviour of an aggregate of such grains can be predicted.

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